The previous two examples suggest that an operator whose characteristic polynomial splits is diagonalizable if and only if the dimension of each eigenspace is equal to the multiplicity of the corresponding eigenvalue. This is indeed true, as we now show. We ~~begin~~ begin with the following lemma, which is a slight variation of Theorem 4.13.2

Lemma 4.13.6 Let T be a linear operator, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. For each $i=1,2,\ldots,k$, let $v_i \in E_{\lambda_i}$. If $v_1+v_2+\cdots+v_k=\theta$ then $v_i=\theta$ for all i.

Proof: Suppose otherwise. So, without loss of generality, we may say $v_i \neq \theta$, for $i=1,2,\ldots,m$ and $v_i=\theta$ for $i=m+1,\ldots,k$

Then for each $i \le m$, v_i is an eigenvector of T corresponding to λ_i and

$$v_1+v_2+\cdots+v_m=\theta$$

But, this contradicts Theorem 4.13.2, which states that $\{v_1, v_2, \ldots, v_m\}$ is linearly independent. So, we conclude that $v_i=\theta$ for all i.

Theorem 4.13.7 Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. For each $i=1,2,\ldots,k$, let S_i be a finite linearly independent subset of the eigenspace E_{λ_i}. Then $S=S_1 \cup S_2 \cup \cdots \cup S_k$ is a linearly independent subset of V.

Proof: Suppose that for each i, $S_i=\{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$.

Then $S = \{ v_{ij} : j=1,2,\ldots,n_i, \ i=1,2,\ldots,k \}$. Consider the scalars a_{ij}, $j=1,2,\ldots n_i$, $i=1,2,\ldots,k$ such that

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} v_{ij} = \theta \quad \cdots \quad (1)$$

For each i, let $w_i = \sum_{j=1}^{n_i} a_{ij} v_{ij}$

Then $w_i \in E_{\lambda_i}$ for each i and from (1)

$w_1 + w_2 + \cdots + w_k = \theta$. So, by Lemma 4.13.6,

$w_i = \theta$ for all i. But each S_i is linearly independent. So, $w_i = \theta$ implies $a_{ij} = 0$ for all j

As $w_i = \theta$ for all i. So, $a_{ij} = 0$ for all i and j

Hence S is linearly independent.

Theorem 4.13.7 tells us how to construct a linearly independent subset of eigenvectors, namely, by collecting bases for the individual eigenspaces. The next theorem tells us when the resulting set is a basis for the entire space.

Theorem 4.13.8 Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial of T splits. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T. Then

(a) T is diagonalizable if and only if the multiplicity of λ_i is equal to $\dim(E_{\lambda_i})$ for all i

(b) If T is diagonalizable and β_i is an ordered basis for E_{λ_i} for each i, then $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for V consisting of eigenvectors of T.

Proof: For each i, let m_i denote the multiplicity of λ_i, $d_i = \dim(E_{\lambda_i})$ and $n = \dim(V)$

First, suppose that T is diagonalizable, Let β be a basis for V consisting of eigenvectors of T. For each i, let $\beta_i = \beta \cap E_{\lambda_i}$, the set of vectors in β that are eigenvectors corresponding to λ_i and let n_i denotes the number of vectors in β_i. Then $n_i \leq d_i$ for each i because β_i is a linearly independent subset of a subspace of dimension d_i and $d_i \leq m_i$ by Theorem 4.12.2.

So, $\sum_{i=1}^{k} n_i = n$ because β contains n vectors.

Also $\sum_{i=1}^{k} m_i = n$ because the degree of the characteristic polynomial of T is equal to the sum of the multiplicities of the eigenvalues.

So, $n = \sum_{i=1}^{k} n_i \leq \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} m_i = n$

It follows that $\sum_{i=1}^{k} (m_i - d_i) = 0$

Since $m_i - d_i \geq 0$ for all i, we conclude that

$m_i = d_i$ for all i

Conversely, suppose that $m_i = d_i$ for all i. We simultaneously show that T is diagonalizable and prove (b). For each i, let β_i be an ordered basis for E_{λ_i} and let $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$. By

Theorem 4.13.7, β is linearly independent. Furthermore, $d_i = m_i$ for all i. So, $\Sigma d_i = \Sigma m_i = n$. So, β is a linearly independent set in V contain n elements which is $\dim(V)$. So, β is an ordered basis for V consisting of eigenvectors of V, and we conclude that T is diagonalizable.

This theorem completes our study of the diagonalization problem. We now ~~summarize~~ summarize our results.

<u>Test for diagonalization</u> Let T be a linear operator on an n dimensional vector space V. Then T is diagonalizable if and only if both the following conditions hold:

1. The characteristic polynomial of T splits.
2. For each eigenvalue λ of T, the multiplicity of λ equals $n - \text{rank}(T - \lambda I)$

These same conditions can be used to test if a square matrix is diagonalizable because the diagonalizability of A is equivalent to diagonalizability of the operator L_A (i.e. $L_A(x) = Ax$)

If T be a diagonalizable operator and $\beta_1, \beta_2, \ldots, \beta_K$ are ordered bases for the eigenspaces of T, then the union $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_K$ is an ordered basis for V consisting of eigenvectors of T and hence $[T]_\beta$ is a diagonal matrix.