the eigenspaces E_{λ_1} and E_{λ_2} respectively. So,

$$\gamma = \gamma_1 \cup \gamma_2 = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$ is an ordered basis for $\mathbb{R}^2$,

consisting of eigenvectors of A. Let

$Q = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$ be the matrix whose columns are the vectors in γ. Then, ~~together with corollary to Theorem~~

$$Q^{-1} A Q = D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

To find A^n for any positive integer n, observe that $A = QDQ^{-1}$. So,

$$A^n = (QDQ^{-1})^n = (QDQ^{-1})(QDQ^{-1})\cdots(QDQ^{-1}) = QD^nQ^{-1}$$

$$= Q\begin{pmatrix} 1^n & 0 \\ 0 & 2^n \end{pmatrix} Q^{-1} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2-2^n & 2-2^{n+1} \\ -1+2^n & -1+2^{n+1} \end{pmatrix}.$$

2.14 Invariant subspaces and the Cayley-Hamilton Theorem

Definition 2.14.1 Let T be a linear operator on a vector space V. A subspace W of V is called a T-invariant subspace of V if $T(W) \subseteq W$, that is, if $T(v) \in W$ for all $v \in W$.

Example 1 Suppose that T is a linear operator on a vector space V. Then the following subspaces of V are T-invariant:

1. $\{\theta\}$
2. V
3. $R(T)$
4. $N(T)$
5. E_λ, for any eigenvalue of T. (Prove it)

Example 2 Let T be a linear operator on $\mathbb{R}^3$ defined by

$T(a, b, c) = (a+b, b+c, 0)$

Then the xy-plane $= \{(x, y, 0) : x, y \in \mathbb{R}\}$ and the x-axis $= \{(x, 0, 0) : x \in \mathbb{R}\}$ are T-invariant subspaces of $\mathbb{R}^3$.

Definition 2.14.2. Let T be a linear operator on a vector space V, and let x be a non-zero vector in V. The subspace $W = \text{span}(\{x, T(x), T^2(x), \dots\})$ is called the T-cyclic subspace of V generated by x. It can be verified (verify it) that W is T-invariant. In fact, W is the "smallest" T-invariant subspace of V containing x. That is, any T-invariant subspace of V containing x must also contain W (Exercise).

Cyclic subspaces have various uses. We apply them in this section to establish Cayley-Hamilton Theorem.

Example 3 Let T be the linear operator on $\mathbb{R}^3$ defined by

$T(a, b, c) = (-b+c, a+c, 3c)$.

We determine the T-cyclic subspace generated by $e_1 = (1, 0, 0)$. Since $T(e_1) = T(1, 0, 0) = (0, 1, 0) = e_2$ and $T^2(e_1) = T(T(e_1)) = T(e_2) = (-1, 0, 0) = -e_1$, it follows that $\text{span}(\{e_1, T(e_1), T^2(e_1), \dots\}) = \text{span}(\{e_1, e_2\})$

$= \{(s, t, 0) : s, t \in \mathbb{R}\}$

Example 4 Let T be the linear operator on $P(\mathbb{R})$

defined by $T(f(x)) = f'(x)$. Then the T-cyclic subspace generated by x^2 is span$(\{x^2, x^2, 2\}) = P_2(\mathbb{R})$.

The existence of a T-invariant subspace provides the opportunity to define a linear operator whose domain is this subspace. If T is a linear operator on V and W is a T-invariant subspace of V, then the restriction T_W of T to W is a mapping from W to W, and it follows that T_W is a linear operator on W (Exercise). As a linear operator, T_W inherits certain properties from its parent operator T. The following result illustrates one way in which the two operators are linked.

Theorem 2.14.3 Let T be a linear operator on a finite dimensional vector space, and let W be a T-invariant subspace of V. Then the characteristic polynomial of T_W divides the characteristic polynomial of T.

Proof: Choose an ordered basis $\gamma = \{v_1, v_2, \dots, v_k\}$ of W and extend it to an ordered basis $\beta = \{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$ of V. Let $A = [T]_\beta$ and $B_1 = [T_W]_\gamma$. Then A can be written in the form $A = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix}$.

Let $f(t)$ be characteristic polynomial of T and $g(t)$ be the characteristic polynomial of T_W. Then

$$f(t) = \det(A - tI_n) = \det\begin{pmatrix} B_1 - tI_k & B_2 \\ O & B_3 - tI_{n-k} \end{pmatrix}$$

$$= g(t)\cdot \det(B_3 - tI_{n-k})$$. So, $g(t)$ divides $f(t)$.

Example 5 Let T be a linear operator on $\mathbb{R}^4$ defined by

$T(a,b,c,d) = (a+b+2c-d,\ b+d,\ 2c-d,\ c+d)$

and let $W = \{(t,s,0,0) : t,s \in \mathbb{R}\}$. Observe that W is a T-invariant subspace of $\mathbb{R}^4$ because for any vector $(a,b,0,0) \in \mathbb{R}^4$

$$T(a,b,0,0) = (a+b,\ b,\ 0,\ 0) \in W$$

Let $\gamma = \{e_1, e_2\}$ where $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$. Then γ is an ordered basis for W. Extend γ to the standard ordered basis β of $\mathbb{R}^4$.

Then $B_1 = [T_W]_\gamma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

and $A = [T]_\beta = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ in the notation of Theorem 2.14.3. Let $f(t)$ be the characteristic polynomial of T and $g(t)$ be the characteristic polynomial of T_W. Then

$$f(t) = \det(A - tI_4) = \det\begin{pmatrix} 1-t & 1 & 2 & -1 \\ 0 & 1-t & 0 & 1 \\ 0 & 0 & 2-t & -1 \\ 0 & 0 & 1 & 1-t \end{pmatrix}$$

$$= \det\begin{pmatrix} 1-t & 1 \\ 0 & 1-t \end{pmatrix}\cdot \det\begin{pmatrix} 2-t & -1 \\ 1 & 1-t \end{pmatrix}$$

$$= g(t)\cdot \det\begin{pmatrix} 2-t & -1 \\ 1 & 1-t \end{pmatrix}$$. So $\ldots$