In view of Theorem 2.14.3, we may use the characteristic polynomial of T_W to gain information about characteristic polynomial of T itself. In this regard, cyclic subspaces are useful because the characteristic polynomial of the restriction of a linear operator ~~of~~ T to a cyclic subspace is readily computable.

Theorem 2.14.4 Let T be a linear operator on a finite dimensional vector space V, and let W denote the T-cyclic subspace of V generated by a non-zero vector ~~in~~ $v \in V$. Let $k = \dim(W)$. Then

(a) $\{v, T(v), T^2(v), \ldots, T^{k-1}(v)\}$ is a basis for W.

(b) If $a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) + \cancel{a_k} T^k(v) = \theta$, then the characteristic polynomial of T_W is

$$f(t) = (-1)^k (a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k).$$

Proof: (a) Since $v \neq \theta$, the set $\{v\}$ is linearly independent. Let j be the largest positive integer for which

$\beta = \{v, T(v), \ldots, T^{j-1}(v)\}$ is linearly independent.

Such a j must exist because V is finite dimensional. Let $Z = \text{span}(\beta)$. Then β is a basis for Z. Furthermore, $T^j(v) \in Z$. We use this information to show that Z is a T-invariant subspace of V. Let $w \in Z$. So, $\exists$ scalars $b_0, b_1, \ldots, b_{j-1}$ such that

$w = b_0 v + b_1 T(v) + \cdots + b_{j-1} T^{j-1}(v)$, and hence

$T(w) = b_0 T(v) + b_1 T^2(v) + \cdots + b_{j-1} T^j(v)$. Thus $T(w)$

is a linear combination of vectors in Z and hence belongs to Z. So, Z is T-invariant. Furthermore $v \in V$. As W is the smallest T-invariant subspace of V that contains v, so that $W \subseteq Z$, Clearly $Z \subseteq W$. and so we conclude that $Z = W$. It follows that β is a basis for W and therefor $\dim(W) = j$ Thus $j = k$. This proves (a).

(b) Now view β (from (a)) as an ordered basis for W. Let $a_0, a_1, \ldots, a_{k-1}$ be the scalars such that

$$a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) + T^k(v) = \theta$$

Observe that $$[T_W]_\beta = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{bmatrix}$$

Which has the characteristic polynomial

$$f(t) = (-1)^k (a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k)$$

Thus $f(t)$ is the characteristic polynomial of T_W, proving (b).

<u>Example 6</u> Let T be the linear operator of Example 3 and let $W = \text{span}(\{e_1, e_2\})$, the T-cyclic subspace generated by e_1. We compute the characteristic polynomial $f(t)$ of T_W in two ways: by means of Theorem 2.14.4 and by means of determinant.

(a) By means of Theorem 2.14.4: From example 3, we have that $\{e_1, e_2\}$ is a cycle that generates W and that $T^2(e_1) = -e_1$. Hence

$$1e_1 + 0T(e_1) + T^2(e_1) = 0$$

So, by Theorem 2.14.4,

$$f(t) = (-1)^2 (1 + 0.t + t^2) = t^2 + 1$$

(b) By means of determinant: Let $\beta = \{e_1, e_2\}$, which is an ordered basis for W. Since $T(e_1) = e_2$ and $T(e_2) = -e_1$, we have

$$[T_W]_\beta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and therefore $f(t) = \det \begin{bmatrix} 0-t & -1 \\ 1 & 0-t \end{bmatrix} = t^2 + 1$

Note: If $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$ be a member of $P_n(F)$ and T be an linear operator on V a vector space over the field F, the we define

$f(T) = a_0 T^n + a_1 T^{n-1} + \cdots + a_1 T + a_0 I$, I is the identity operator on V. So, $f(T)$ is a linear operator on V.

As an illustration of importance of Theorem 2.14.4 we prove Cayley-Hamilton Theorem:

Theorem 2.14.5 (Cayley-Hamilton Theorem) Let T be a linear operator on a finite dimensional vector space V, and let $f(t)$ be the characteristic polynomial of of T. Then $f(T) = T_0$, the zero transformation. That is, T "satisfies" the characteristic equation.

Proof : We show that $f(T)(v) = \theta$ for all $v \in V$.
This is true if $v = \theta$ as $f(T)$ is linear; so suppose $v \neq \theta$. Let W be the T-cyclic subspace generated by v and suppose that $\dim(W) = k$. By Theorem 2.14.4 (a), there exists scalars $a_0, a_1, a_2, \ldots, a_{k-1}$ such that

$$a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) + T^k(v) = \theta$$

Hence Theorem 2.14.4 (b) implies that

$$g(t) = (-1)^k \left(a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k\right)$$

is the characteristic polynomial of T_W.
Combining these two equations gives

$$g(T)(v) = (-1)^k \left(a_0 I + a_1 T + \cdots + a_{k-1} T^{k-1} + T^k\right)(v) = \theta$$

By theorem 2.14.3, $g(t)$ divides $f(t)$; hence there exists a polynomial $q(t)$ such that $f(t) = q(t)\, g(t)$

So, $f(T)(v) = q(T)\, g(T)(v) = q(T)\big(g(T)(v)\big) = q(T)(\theta) = \theta$.

<u>Example 7.</u> Let T be a linear operator on $\mathbb{R}^2$ defined by $T(a,b) = (a+2b, -2a+b)$, and let $\beta = \{e_1, e_2\}$.

Then $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

where $A = [T]_\beta$. The characteristic polynomial of T is, therefore, $f(t) = \det(A - tI) = \det\begin{pmatrix} 1-t & 2 \\ -2 & 1-t \end{pmatrix}$
$= t^2 - 2t + 5$. It can be easily verified that $T_\theta = f(T) = T^2 - 2T + 5I$. Similarly, $\cancel{f(A) = A^2 - 5A}$