$$f(A) = A^2 - 2A + 5I = \begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix} + \begin{pmatrix} -2 & -4 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Corollary 2.14.6 (Cayley-Hamilton Theorem for matrices). Let A be an $n \times n$ matrix and let $f(t)$ be the characteristic polynomial of A. Then $f(A) = O$, the $n \times n$ zero matrix.

Proof: Exercise

2.15 The minimal polynomials

The Cayley-Hamilton theorem tells us that for any linear operator T on an ~~n~~ n-dimensional vector space, there is a polynomial $f(t)$ of degree n such that $f(T) = T_0$, namely, the characteristic polynomial of T. Hence there is a polynomial of least degree with this property, and this degree is at most n. If $g(t)$ is such a polynomial, we can divide $g(t)$ by its leading coefficient to obtain another polynomial $p(t)$ of the same degree with leading coefficient 1, that is, $p(t)$ is a monic polynomial.

Definition 2.15.1 Let T be a linear operator on a finite dimensional vector space. A polynomial $p(t)$ is called a minimal polynomial of T if $p(t)$ is a monic polynomial of least positive degree for which $p(T) = T_0$

(NOTE: A monic polynomial is a polynomial in which the leading coefficient i.e, the non-zero coefficient of highest degree is equal to 1)

Theorem 2.15.2 Let $p(t)$ be a ~~minimal~~ minimal polynomial of a linear operator T on a finite dimensional vector space V. Then,

(a) For any polynomial $g(t)$, if $g(T) = T_0$, then $p(t)$ divides $g(t)$. In particular, $p(t)$ divides the characteristic polynomial of T.

(b) The minimal polynomial of T is unique

Proof (a) Let $g(t)$ be a polynomial for which $g(T) = T_0$. By the division algorithm for polynomials, there exists polynomials $q(t)$ and $r(t)$ such that

$$g(t) = q(t)p(t) + r(t) \quad \cdots (1)$$

where $r(t)$ has degree less than the degree of $p(t)$. Substituting T in (1) and using that $g(T) = p(T) = T_0$, we have $r(T) = T_0$. Since $r(t)$ has degree less than $p(t)$ and $p(t)$ is the minimal polynomial of T, $r(t)$ must be zero polynomial. So, (1) simplies to $g(t) = q(t)p(t)$, proving (a)

(b) Suppose that $p_1(t)$ and $p_2(t)$ are each minimal polynomials of T. Then $p_1(t)$ divides $p_2(t)$ by (a) Since $p_1(t)$ and $p_2(t)$ have the same degree, we have that $p_2(t) = c p_1(t)$ for some scalar c. Because $p_1(t)$ and $p_2(t)$ are monic polynomial, $c = 1$; hence $p_1(t) = p_2(t)$.

The minimal polynomial of a linear operator has an obvious analog for a matrix.

Definition 2.15.3 Let $A \in M_{n\times n}(F)$. The minimal polynomial $p(t)$ of A is the monic polynomial of least positive degree for which $p(A) = O$.

The following results are now immediate.

Theorem 2.15.4 Let T be a linear operator on a finite dimensional vector space V, and let β be an ordered basis for V. Then the minimal polynomial of T is the same as the minimal polynomial of $[T]_\beta$.

Proof: Exercise

Corollary 2.15.5 For any $A \in M_{n\times n}(F)$, the minimal polynomial of A is the same as minimal polynomial of L_A.

Proof: Exercise

Theorem 2.15.6 Let T be a linear operator on a finite dimensional vector space V, and let $p(t)$ be the minimal polynomial of T. A scalar λ is eigenvalue of T if and only if $p(\lambda) = 0$. Hence the characteristic polynomial and the minimal polynomial have the same zeros.

Proof: Let $f(t)$ be the characteristic polynomial of T. Since $p(t)$ divides $f(t)$, $\exists$ a polynomial $q(t)$ such that $f(t) = q(t)\,p(t)$. If λ is a zero of $p(t)$, then $f(\lambda) = q(\lambda)\,p(\lambda) = q(\lambda)\cdot 0 = 0$. So, λ is a zero of $f(t)$; that is, λ is an eigenvalue of T.

Conversely, suppose that λ is an eigenvalue of T, and let $x \in V$ be an eigenvector corresponding to

λ. So, we have

$$\theta = T_{\theta}(x) = p(T)(x) = p(\lambda)x. \quad [\text{As } T(x) = \lambda x]$$

Since, $x \neq \theta$, it follows that $p(\lambda) = 0$ and so λ is a zero of $p(t)$.

The following corollary is immediate

Corollary 2.15.7. Let T be a linear operator on a finite dimensional vector space V with minimal polynomial $p(t)$ and characteristic polynomial $f(t)$. Suppose that $f(t)$ factors as

$$f(t) = (\lambda_1 - t)^{n_1}(\lambda_2 - t)^{n_2} \cdots (\lambda_k - t)^{n_k}$$

~~where $\lambda_1, \lambda_2, \ldots \lambda_k$~~
Where $\lambda_1, \lambda_2, \ldots \lambda_k$ are distinct eigenvalues of T. Then $\exists$ integers $m_1, m_2, \ldots, m_k$ such that $1 \leq m_i \leq n_i$ for all i and

$$p(t) = (t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k}$$

~~Exa~~ <u>Example 1</u>

We compute the minimal polynomial of the matrix

$$A = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix}$$

Since A has the characteristic polynomial

$$f(t) = \det \begin{pmatrix} 3-t & -1 & 0 \\ 0 & 2-t & 0 \\ 1 & -1 & 2-t \end{pmatrix} = -(t-2)^2(t-3),$$

the minimal polynomial of A must be either $(t-2)(t-3)$ or $(t-2)^2(t-3)$ by corollary 2.15.7. Substituting A into $p(t) = (t-2)(t-3)$, we find $p(A) = 0$; hence $p(t)$ is the minimal polynomial of A.