Example 5 Let $A \in M_{n\times n}(\mathbb{R})$ satisfy $A^3 = A$. We show that A is diagonalizable. Let $g(t) = t^3 - t = t(t-1)(t+1)$. Then $g(A) = O$ and hence the minimal polynomial $p(t)$ of A divides $g(t)$. Since $g(t)$ has no repeated factors, neither does $p(t)$. Thus A is diagonalizable by Theorem 2.15.9

Example 6 In example 3, we saw that the minimal polynomial of the differential operator D on $P_2(\mathbb{R})$ is t^3. Hence by Theorem 2.15.9, D is not diagonalizable.

3. Canonical Forms (Jordan & Rational)

As we have learned earlier, the advantage of a diagonalizable operator lies in the simplicity of its description. Such an operator has a diagonal matrix representation, or, equivalently, there is an ordered basis for the underlying vector space consisting of eigenvectors of the operator. However, not every linear operator is diagonalizable, even if its characteristic polynomial splits. For example, If T be the linear operator on $P_2(\mathbb{R})$ defined by $T(f(x)) = f'(x)$, then we have seen earlier that its characteristic polynomial splits but it is not diagonalizable.

It is the purpose of this section to consider alternative matrix representations for non-diagonalizable operators. These representations are called canonical forms. There are different kinds of canonical ~~forms~~ forms, and their advantages and disadvantages depend on how they are applied. The choice of a canonical form is determined by the appropriate choice of an ordered basis. Naturally, the canonical forms of a linear operator are not diagonal matrices if the linear operator is not diagonalizable.

In this section, we treat two common canonical forms. The first of these, the ~~Jordan~~ Jordan canonical form, requires that the characteristic polynomial of the operator splits. This form is always available if the underlying field is algebraically ~~cl~~ closed, that is, if every polynomial with coefficients from the field splits. For example, the field of complex numbers is algebraically closed by the fundamental theorem of algebra. We describe Jordan canonical form in two parts. Then we treat rational canonical form which does not require such a factorization of the characteristic polynomial.

7.1 The Jordan Canonical form (part-1)

Let T be a linear operator on a finite dimensional vector space, and suppose that the characteristic polynomial of T splits. Recall that the diagonalizability of T depends on whether the union of ordered bases for the distinct eigenspaces of T is an ordered basis for V. So a lack of diagonalizability means that at least one eigenspace of T is too "small".

In this section, we extend the definition of eigenspace to generalized eigenspace. From these subspaces, we select ordered bases whose union is an ordered basis β for V such that

$$[T]_\beta = \begin{pmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & & \\ O & O & \cdots & A_k \end{pmatrix}$$

where each O is a zero matrix, and each A_i is a square matrix of the form (λ) or

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & \lambda & 1 \\ 0 & 0 & 0 & & 0 & \lambda \end{pmatrix}$$

for some eigenvalue λ of T. Such a matrix A_i is called a <u>Jordan block</u> corresponding to λ, and matrix $[T]_\beta$ is called a Jordan canonical form of T. We also say that the ordered basis β is a Jordan canonical basis for T. Observe that each Jordan block A_i is "almost" a diagonal matrix - in fact, $[T]_\beta$ is a diagonal matrix if each A_i is of the form (λ).

<u>Example 1</u> Suppose that T is linear operator on $\mathbb{C}^8$, and $\beta = \{v_1, v_2, \ldots, v_8\}$ is an ordered basis

for $\mathbb{C}^8$ ($\mathbb{C}$ is the field of complex numbers) such that

$$J = [T]_\beta = \left(\begin{array}{ccc|c|cc|cc} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

is a Jordan canonical form of T. Notice that the characteristic polynomial of T is $\det(J - tI) = (t-2)^4(t-3)^2 t^2$, and hence the multiplicity of each eigenvalue is the number of times that the eigenvalue appears on the diagonal of J. Also observe that v_1, v_4, v_5 and v_7 are the only vectors in β that are eigenvectors of T. These are the vectors corresponding to the columns of J with number 1 above the diagonal entry.

Now we prove that every linear operator whose characteristic polynomial splits has a Jordan canonical form that is unique up to the order of the Jordan blocks. Nevertheless, it is not the case that the Jordan canonical form is completely determined by the characteristic polynomial of the operator. For example, let T′ be a linear operator on $\mathbb{C}^8$ such that $[T']_\beta$, where β is an ordered basis as in example 1 and

$$J' = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$