$\lambda = -1$ is the only eigenvalue of T, and hence $K_\lambda = P_2(\mathbb{R})$ by Theorem 7.1.6. So β is a basis for K_λ. Now

$$\dim(E_\lambda) = 3 - \text{rank}(A+I) = 3 - \text{rank}\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} = 3-2 = 1$$

Therefore a basis for K_λ can not be a union of two or three cycles because the initial vector of each cycle is an eigenvector, and there do not exist two or more linearly independent eigenvectors. So the desired basis must consist of a single cycle of length 3. If γ is such a cycle, then γ determines a single Jordan block $[T]_\gamma = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ which is a Jordan canonical form of T.

The end vector $h(x)$ of such a cycle must satisfy $(T+I)^2(h(x)) \neq \theta$. In any basis for K_λ, there must be a vector that satisfies this condition, or else no vector in K_λ satisfies this condition, contrary to our reasoning. Testing the vectors in β, we see that $h(x) = x^2$ is acceptable. Therefore

$$\gamma = \{(T+I)^2(x^2), (T+I)(x^2), x^2\} = \{2, -2x, x^2\}$$ is

a Jordan canonical basis for T.

7.2 The Jordan Canonical form (Part-II)

For the purpose of this section, we fix a linear operator T on an n dimensional vector space V such that the characteristic polynomial of T splits. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T.

By Theorem 7.1.12 (page-148), each generalized eigenspace K_{λ_i} contains and ordered basis β_i consisting of a union of disjoint cycles of generated eigenvectors corresponding to λ_i.

So, by Theorems 7.1.6 (b) and 7.1.9 (page-142 and page-145 respectively), the union $\beta = \bigcup_{i=1}^{k} \beta_i$ is a Jordan Canonical basis for T.

For each i, let T_i be the restriction of T to K_{λ_i}, and let $A_i = [T_i]_{\beta_i}$. Then A_i is the the Jordan Canonical form of T_i and

$$J = [T]_\beta = \begin{pmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & A_k \end{pmatrix}$$

is the Jordan Canonical form of T. In this matrix, each O is a zero matrix of appropriate size.

In this section, we compute the matrices A_i and the bases β_i, thereby computing J and β as well. While developing a method for finding J, it becomes evident that in some sense the matrices A_i are unique.

To aid in formulating the uniqueness theorem of J, we adopt the following convention:

The basis β_i for K_{λ_i} will henceforth be ordered in such a way that the cycles appears in order of decreasing length. That is, if β_i is a disjoint union of cycles $\gamma_1, \gamma_2, \ldots, \gamma_{n_i}$ and if the of the cycle γ_j is p_j, we index the cycles so that $p_1 \geq p_2 \geq \cdots \geq p_{n_i}$. This ordering of the cycles limits the possible orderings of vectors in β_i, which in turns determines the matrix A_i. It is in this sence A_i is unique. It then follows that the Jordan canonical form for T is unique upto an ordering of the eigenvalues of T. As we shall see, there is no uniqueness theorem for the bases for β_i or for β. Specifically, we show that for each i, the number n_i of cycles that form β_i, and the length p_j $(j = 1, 2, \ldots, n_i)$ of each cycle, is completely determined by T.

<u>Example 1</u> To illustrate the discussion above, suppose that, for some i, the ordered basis β_i for K_{λ_i} is the union of the cycles $\beta_i = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ with respective length $p_1 = 3$, $p_2 = 3$, $p_3 = 2$ and $p_4 = 1$. Then

$$A_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_i & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_i \end{pmatrix}$$

To help us visualize each of the matrices A_i and ordered bases β_i, we use an array of dots called a dot diagram of T_i, where T_i is the restriction of T to K_{λ_i}. Suppose that β_i is a disjoint union of cycles of generalized eigenvectors $\gamma_1, \gamma_2, \cdots, \gamma_{n_i}$ with lengths $p_1 \geq p_2 \geq \cdots \geq p_{n_i}$, respectively. The dot diagram of T_i contains one dot for each vector in β_i, and the dots are configured according to the following rules: 1. The array consists of n_i columns (one column for each cycle).

2. Counting from left to right, the jth column consists of the p_j dots that correspond to the vectors of γ_j starting with the initial vector at the top and continuing down to the end vector.

Denote the end vectors of the cycles by $v_1, v_2, \cdots, v_{n_i}$. In the following dot diagram of T_i, each dot is labeled with the name of the vector in β_i to which it corresponds.

$$
\begin{array}{llll}
\bullet\ (T-\lambda_i I)^{p_1-1}(v_1) & \bullet\ (T-\lambda_i I)^{p_2-1}(v_2) & \cdots & \bullet\ (T-\lambda_i I)^{p_{n_i}-1}(v_{n_i}) \\
\bullet\ (T-\lambda_i I)^{p_1-2}(v_1) & \bullet\ (T-\lambda_i I)^{p_2-2}(v_2) & \cdots & \bullet\ (T-\lambda_i I)^{p_{n_i}-2}(v_{n_i}) \\
\quad \vdots & \quad \vdots & & \quad \vdots \\
 & & & \bullet\ (T-\lambda_i I)(v_{n_i}) \\
 & & & \bullet\ v_{n_i} \\
 & \bullet\ (T-\lambda_i I)(v_2) & \cdots & \\
 & \bullet\ v_2 & \cdots & \\
\bullet\ (T-\lambda_i I)(v_1) & & & \\
\bullet\ v_1 & & &
\end{array}
$$

Notice that the dot diagrams of T_i has n_i columns (one for each cycle) and p_1 rows. Since $p_1 \geq p_2 \cdots \geq p_{n_i}$, the columns of the dot diagram become shorter (or at least not longer) as we move from left to right..