Then the characteristic polynomial of T' is also $(t-2)^4(t-3)^2t^2$. But the operator T' has the Jordan Canonical form J', which is different from J, the Jordan Canonical form of the linear operator T of Example 1.

Consider again the matrix J and the ordered basis β of Example 1. Notice that $T(v_2) = v_1 + 2v_2$ and therefore, $(T-2I)(v_2) = v_1$. Similarly, $(T-2I)(v_3) = v_2$. Since v_1 and v_4 are eigenvectors of T corresponding to $\lambda = 2$, it follows that $(T-2I)^3(v_i) = \theta$ for $i = 1, 2, 3$ and 4. Similarly $(T-3I)^2(v_i) = \theta$ for $i = 5, 6$ and $(T-0I)^2(v_i) = \theta$ for $i = 7, 8$.

Because of the structure of each Jordan block in a Jordan Canonical form, we can generalize these observations: If v lies in a Jordan Canonical basis for a linear operator T and is associated with a Jordan block with diagonal entry λ, then $(T-\lambda I)^p(v) = \theta$ for sufficiently large p. Eigenvectors satisfy the condition for $p = 1$.

Definition 7.1.1 Let T be a linear operator on a vector space V, and let λ be a scalar. A non-zero vector $x \in V$ is called a generalized eigenvector of T corresponding to λ if $(T-\lambda I)^p(x) = \theta$ for some positive integer p.

Notice that if x is a generalized eigenvector of T corresponding to λ and p is the smallest positive integer for which $(T-\lambda I)^p(x) = \theta$, then $(T-\lambda I)^{p-1}(x)$ is an eigenvector of T corresponding to λ. Therefore λ is an eigenvalue of T.

In the context of Example 1, each vector ~~β~~ in β is is a generalized eigenvector of T. In fact, v_1, v_2, v_3 and v_4 correspond to the scalar 2, v_5 and v_6 correspond to the scalar 3, and v_7 and v_8 correspond to the scalar 0.

Just as eigenvectors lie in eigenspaces, generalized eigenvectors lie in "generalized eigenspaces"

Definition 7.1.2 Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. The generalized eigenspace of T corresponding to λ, denoted by K_λ, is the subset of V defined by $K_\lambda = \{x \in V : (T-\lambda I)^p(x) = \theta$ for some positive integer $p\}$. Note that K_λ consists of zero vector and all generalized eigenvectors corresponding to λ.

Recall that a subspace W of V is T-invariant for a linear operator T if $T(W) \subseteq W$. We that for any polynomial $g(x)$, if W is T-invariant, then it is also $g(T)$-invariant. Furthermore, the range of a linear operator T is T-invariant.

Theorem 7.1.3. Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. Then
(a) K_λ is a T-invariant subspace of V containing E_λ (the eigenspace of T corresponding to λ).
(b) For any scalar $\mu \neq \lambda$, the restriction of $T-\mu I$ to K_λ is one-to-one.

Proof: (a) Clearly, $\theta \in K_\lambda$. Suppose that x and y are in K_λ. Then there exist positive integers p and q such that

$$(T-\lambda I)^p(x) = (T-\lambda I)^q(y) = \theta.$$

Therefore,

$$\begin{aligned}(T-\lambda I)^{p+q}(x+y) &= (T-\lambda I)^{p+q}(x) + (T-\lambda I)^{p+q}(y) \\ &= (T-\lambda I)^q(\theta) + (T-\lambda I)^p(\theta) \\ &= \theta\end{aligned}$$

and hence $x+y \in K_\lambda$. The proof that K_λ is closed under scalar multiplication is straightforward (prove it).

To show that K_λ is T-invariant, consider any $x \in K_\lambda$. Choose a positive integer p such that $(T-\lambda I)^p(x) = \theta$. Then

$(T-\lambda I)^p T(x) = T(T-\lambda I)^p(x) = T(\theta) = \theta$. Therefore $T(x) \in K_\lambda$.

Finally, it is a simple observation that E_λ is contained in K_λ.

(b) Let $x \in K_\lambda$ and $(T-\mu I)(x) = \theta$. By way of contradiction, suppose that $x \neq \theta$. Let p be the smallest integer for which $(T-\lambda I)^p(x) = \theta$, and let $y = (T-\lambda I)^{p-1}(x)$.

Then $(T-\lambda I)(y) = (T-\lambda I)^p(x) = \theta$ and hence $y \in E_\lambda$.

Furthermore, $(T-\mu I)(y) = (T-\mu I)(T-\lambda I)^{p-1}(x)$

$= (T-\lambda I)^{p-1}(T-\mu I)(x) = \theta$ so that $y \in E_\mu$. But $E_\lambda \cap E_\mu = \{\theta\}$, and thus $y = \theta$, contrary to the hypothesis. So, $x = \theta$, and the restriction of $T-\mu I$ to K_λ is one-to-one.

<u>Theorem 7.1.4</u> Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial of T splits. Suppose that λ is an eigenvalue of T with multiplicity m. Then

(a) $\dim(K_\lambda) \leq m$

(b) $K_\lambda = N((T-\lambda I)^m)$.

Proof: (a) Let $W = K_\lambda$ and let $h(t)$ be the characteristic polynomial of T_W. By previous result on characteristic polynomial, $h(t)$ divides the characteristic polynomial of T, and by Theorem 7.1.3(b), λ is the only eigenvalue of T_W. Hence $h(t) = (-1)^d (t-\lambda)^d$, where $d = \dim(W)$, and $d \le m$.

(b) Clearly $N((T-\lambda I)^m) \subseteq K_\lambda$. Now let W and $h(t)$ be as in (a). Then $h(T_W)$ is identically zero by the Cayley-Hamilton theorem; therefore $(T-\lambda I)^d(x) = \theta$ for all $x \in W$. Since $d \le m$, we have $K_\lambda \subseteq N(T-\lambda I)^m$.

<u>Theorem 7.1.5</u> Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial of T splits, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T. Then, for every $x \in V$, there exists vectors $v_i \in K_{\lambda_i}$, $i = 1, 2, \ldots, k$, such that

$$x = v_1 + v_2 + \cdots + v_k$$

Proof: The proof is by mathematical induction on the number k of distinct eigenvalues of T. First suppose that $k=1$, let m be the multiplicity of λ_1. Then $(\lambda_1 - t)^m$ is the characteristic polynomial of T, hence $(\lambda_1 I - T)^m = T_\theta$ by the Cayley-Hamilton theorem. Thus $V = K_{\lambda_1}$, and the result follows.

Now suppose that for some integer $k > 1$, the result is established whenever T has fewer than k distinct eigenvalues, and suppose that T has k distinct eigenvalues. Let m be multiplicity of λ_k, and let $f(t)$ be the characteristic polynomial of T. Then $f(t) = (t-\lambda_k)^m g(t)$ for some polynomial $g(t)$ not divisible by $(t-\lambda_k)$. Let $W = R((T-\lambda_k I)^m)$. Clearly W is T-invariant. Observe that $(T-\lambda_k I)^m$ maps K_{λ_i} onto itself for $i < k$. For, suppose that