$i<k$. Since $(T-\lambda_k I)^m$ maps K_{λ_i} into itself and $\lambda_k \neq \lambda_i$, the restriction of $T-\lambda_k I$ to K_{λ_i} is one-to-one (by Theorem 7.1.3 (b)) and hence is onto. One consequence of this is that for $i<k$, K_{λ_i} is contained in W; hence λ_i is an eigenvalue of T_W for $i<k$.

Next, observe that λ_k is not an eigenvalue of T_W. For, suppose that $T(v) = \lambda_k v$ for some $v \in W$. Then $v = (T-\lambda_k I)^m(y)$ for some $y \in V$, it follows that

$$\theta = (T-\lambda_k I)(v) = (T-\lambda_k I)^{m+1}(y).$$

Therefore $y \in K_{\lambda_k}$. So, by Theorem 7.1.4, $v = (T-\lambda_k I)^m(y) = \theta$.

Since every eigenvalue of T_W is an eigenvalue of T, the distinct eigenvalues of T_W are $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$.

Now let $x \in V$. Then $(T-\lambda_k I)^m(x) \in W$. Since T_W has the $k-1$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$, the induction hypothesis applies. Hence there are vectors $w_i \in K'_{\lambda_i}$, $i = 1, 2, \ldots, k-1$, such that

$$(T-\lambda_k I)^m(x) = w_1 + w_2 + \cdots + w_{k-1}$$ (K'_{λ_i} is the generalised eigenspace corresponding T_W)

Since $K'_{\lambda_i} \subseteq K_{\lambda_i}$ for $i<k$ and $(T-\lambda_k I)^m$ maps K_{λ_i} onto itself for $i<k$, there exists vectors $v_i \in K_{\lambda_i}$ such that $(T-\lambda_k I)^m(v_i) = w_i$ for $i<k$. Thus we have

$$(T-\lambda_k I)^m(x) = (T-\lambda_k I)^m(v_1) + (T-\lambda_k I)^m(v_2) + \cdots + (T-\lambda_k I)^m(v_{k-1})$$

and it follows that $x - (v_1 + v_2 + \cdots + v_{k-1}) \in K_{\lambda_k}$. So, there exists a vector $v_k \in K_{\lambda_k}$ such that

$x = v_1 + v_2 + \cdots + v_k$.

<u>Theorem 7.1.6</u> Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial of T splits, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T with corresponding multiplicities $m_1, m_2, \ldots, m_k$. For $i = 1, 2, \ldots, k$, let β_i be an ordered basis for K_{λ_i}. Then the following statements are true:

(a) $\beta_i \cap \beta_j = \phi$ for $i \neq j$

(b) $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for V.

(c) $\dim(K_{\lambda_i}) = m_i$ for all i.

Proof: (a) Suppose that $x \in \beta_i \cap \beta_j \subseteq K_{\lambda_i} \cap K_{\lambda_j}$, where $i \neq j$. By Theorem 7.1.3(b), $T - \lambda_i I$ is one-to-one on K_{λ_j}, and therefore $(T - \lambda_i I)^p(x) \neq \theta$ for any positive integer p. But this contradicts the fact that $x \in K_{\lambda_i}$, and the result follows.

(b) Let $x \in V$. By Theorem 7.1.5, for $i = 1, 2, \ldots, k$, there exist vectors $v_i \in K_{\lambda_i}$, such that $x = v_1 + v_2 + \cdots + v_k$. Since each v_i is a linear combination of the vectors of β_i, it follows that x is a linear combination of the vectors of β. Therefore β spans V. Let q be the number of vectors in β. Then $\dim(V) \leq q$. For each i, let $d_i = \dim(K_{\lambda_i})$. Then by Theorem 7.1.4(a),

$$q = \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} m_i = \dim(V) \quad . \text{ Hence } q = \dim(V).$$

Consequently β is a basis for V.

(c) Using the notation and the result of (b), we see that $\sum_{i=1}^{k} d_i = \sum_{i=1}^{k} m_i$. But $d_i \le m_i$ by Theorem 7.1.4(a), and therefore $d_i = m_i$ for all i.

Corollary 7.1.7 Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial of T splits. Then T is diagonalizable if and only if $E_\lambda = K_\lambda$ for every eigenvalue λ of T.

Proof: Combining Theorem 7.1.6 and ~~Theorem 4.13.8 (page-110)~~ Theorem 4.13.8(a) (page-110), we see that T is diagonalizable if and only if $\dim(E_\lambda) = \dim(K_\lambda)$ for each eigenvalue λ of T. But $E_\lambda \subseteq K_\lambda$, and hence these subspaces have the same dimension if and only if they are equal.

We now focus our attention on the problem of selecting suitable bases for the generalised eigenspaces of a linear operator so that we may use Theorem 7.1.6 to obtain a Jordan canonical basis for the operator. For this ~~pur~~ purpose, we consider again the basis β of Example 1. We have seen that the first four vectors of β lie in the generalized eigenspace K_2. Observe that the vectors in β that determine the first Jordan block of J are of the form

$$\{v_1, v_2, v_3\} = \{(T-2I)^2(v_3), (T-2I)(v_3), v_3\}.$$

Furthermore, observe that $(T-2I)^3(v_3) = \theta$. The relation between these vectors is the key to finding Jordan canonical basis. These leads to the following definition.

<u>Definition 7.1.8</u> Let T be a linear operator on a vector space V, and let x be a generalised eigenvector of T corresponding to the eigenvalue λ. Suppose that p is the smallest positive integer for which $(T-\lambda I)^p(x) = \theta$. Then the ordered set

$$\{(T-\lambda I)^{p-1}(x), (T-\lambda I)^{p-2}(x), \ldots, (T-\lambda I)(x), x\}$$

is called a <u>cycle of generalised eigenvectors of T corresponding to λ</u>. The vector $(T-\lambda I)^{p-1}(x)$ and x are called the <u>initial vector</u> and the <u>end vector</u> of the cycle, respectively. We say that the <u>length</u> of the cycle is p.

Notice that the initial vector of a cycle of generalized eigenvectors of a linear operator T is the only eigenvector of T in the cycle. Also observe that if x is an eigenvector of T corresponding to the eigenvalue λ, then the set $\{x\}$ is a cycle of generalized eigenvectors of T corresponding to λ of length 1.

In example 1, the subsets $\beta_1 = \{v_1, v_2, v_3\}$, $\beta_2 = \{v_4\}$, $\beta_3 = \{v_5, v_6\}$ and $\beta_4 = \{v_7, v_8\}$ are the cycles of the generalized eigenvectors of T that occur in β. Notice that β is a disjoint union of these cycles. Furthermore, setting $W_i = span(\beta_i)$ for $i=1,2,3,4$, we see that β_i is a basis for W_i and $[T_{W_i}]_{\beta_i}$ is