the ith Jordan block of the Jordan Canonical form of T. This is precisely the condition that is required for a Jordan Canonical basis.

Theorem 7.1.9. Let T be a linear operator on a finite dimensional vector space V whose characteristic polynomial splits, and suppose that β is a basis for V such that β is a disjoint union of cycles of generalized eigenvectors of T. Then the following statements are true:

(a) For each cycle γ of generalized eigenvectors contained in β, $W = span(\gamma)$ is T-invariant, and $[T_W]_\gamma$ is a Jordan block.

(b) β is a Jordan Canonical basis for V.

Proof (a) Suppose that γ corresponds to λ and γ has length p, and x is the end vector of γ. Then

$$\gamma = \{v_1, v_2, \ldots, v_p\} \text{ where } v_i = (T-\lambda I)^{p-i}(x) \text{ for } i<p$$
$$\text{and } v_p = x.$$

So, $(T-\lambda I)(v_1) = (T-\lambda I)^p(x) = 0$ and hence $T(v_1) = \lambda v_1$.

For $i>1$ $\quad (T-\lambda I)(v_i) = (T-\lambda I)^{p-(i-1)}(x) = v_{i-1}$

Therefore T maps W into itself, and by the preceding equations, we see that $[T_W]_\gamma$ is a Jordan block.

For (b), simply repeat the arguments of (a) for each cycle in β in order to obtain $[T]_\beta$ (write it).

<u>Theorem 7.1.10</u> Let T be a linear operator on a vector space V, and λ be an eigenvalue of T. Suppose that $\gamma_1, \gamma_2, \ldots, \gamma_q$ are cycles of generalised eigenvectors of T corresponding to λ such that the initial vectors of the γ_i's are distinct and form a linearly independent set. Then the γ_i's are disjoint, and their union $\gamma = \bigcup_{i=1}^{q} \gamma_i$ is linearly independent.

Proof: ~~If~~ If possible, let $x \in \gamma_i \cap \gamma_j$ for some ~~i, j~~ i, j, $i \neq j$, then we may find ~~positive integer p such that~~ the smallest positive integer p such that $(T-\lambda I)^p(x) = \theta$. This implies that the initial eigenvectors of γ_1 and γ_2 are both $(T-\lambda I)^{p-1}(x)$, a contradiction. Hence all the cycles are disjoint.

For the proof that γ is linear independent, we apply mathematical induction on the number of vectors in γ. If this ~~the~~ number is less than 2, then the result is clear (prove it). So, assume that, for some integer $n > 1$, the result is valid when γ has fewer than n vectors and suppose that γ has exactly n vectors. Let W be the subspace of V generated by γ. Clearly W is $(T-\lambda I)$-invariant, and $\dim(W) \leq n$. Let U denote the restriction of

$(T-\lambda I)$ to W. For each i, let γ_i' denote the cycle obtained from γ_i by deleting the end vector. Note that if γ_i has length one, then $\gamma_i' = \phi$. In the case $\gamma_i' \neq \phi$ each vector of γ_i' is the image under U of a vector in γ_i and conversely, every non-zero image under U of a vector of γ_i is contained in γ_i'. Let $\gamma_i' = \bigcup_i \gamma_i'$

Then by the last statement, γ' generates $R(U)$.

Furthermore, γ' consists of $n-q$ vectors and the initial vectors of γ_i''s are also initial vectors of γ_i's. Thus we may apply induction hypothesis to conclude that γ' is linearly independent.

Therefore γ' is a basis for $R(U)$. Hence $\dim(R(U)) = n-q$. Since the q initial vectors of the γ_i's form a linearly independent set and lies in $N(U)$, we have $\dim(N(U)) \geq q$

From these inequalities and the dimension theorem,

we obtain
$$\begin{aligned} n &\geq \dim(W) \\ &= \dim(R(U)) + \dim(N(U)) \\ &\geq n-q+q \\ &= n \end{aligned}$$

We conclude that $\dim(W) = n$. Since γ generates W and consists of n vectors, it must be a basis for W. Hence γ is linearly independent

Corollary 7.1.11 Every cycle of generalized eigenvectors of a linear operator is linearly independent.

Proof: Exercise.

Theorem 7.1.12 Let T be a linear operator on a finite dimensional vector space V, and let λ be an eigenvalue of T. Then K_λ has an ordered basis consisting of a union of disjoint cycles of generalized eigenvectors corresponding to λ.

Proof: The proof is by mathematical induction on $n = \dim(K_\lambda)$. The result is clear for $n=1$. So, suppose that for some integer $n>1$, the result is valid whenever $\dim(K_\lambda) < n$, and assume that $\dim(K_\lambda) = n$. Let U denote the restriction of $T-\lambda I$ to K_λ. Then $R(U)$ is a subspace of K_λ of lesser dimension, and $R(U)$ is the space of generalized eigenvectors corresponding to λ for the restriction of T to $R(U)$. Therefore, by induction hypothesis, there exist disjoint cycles $\gamma_1, \gamma_2, \ldots, \gamma_q$ of generalized eigenvectors of this restriction, and hence of T itself, corresponding to λ for which $\gamma = \bigcup_{i=1}^{q} \gamma_i$ is a basis for $R(U)$. For $i = 1, 2, \ldots, q$, the end vectors of γ_i is the image under U of a vector $v_i \in K_\lambda$, and so we can extend each γ_i to a larger cycle $\overline{\gamma}_i = \gamma_i \cup \{v_i\}$ of generalized eigenvectors of T corresponding to λ.