For $i = 1, 2, \ldots, q$, let w_i be the initial vector of $\bar{\gamma}_i$ (and hence of γ_i). Since $\{w_1, w_2, \ldots, w_q\}$ is a linearly independent subset of E_λ, this set can be extended to a basis $\{w_1, w_2, \ldots, w_q, u_1, u_2, \ldots, u_s\}$ for E_λ. Then $\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_q, \{u_1\}, \{u_2\}, \ldots, \{u_s\}$ are disjoint cycles of generalized eigenvectors of T corresponding to λ such that the initial vectors of these cycles are linearly independent. Therefore their union $\bar{\gamma}$ is a linearly independent subset of K_λ by Theorem 7.1.10.

We show that $\bar{\gamma}$ is a basis for K_λ. Suppose that γ consists of $r = \text{rank}(U)$ vectors. Then $\bar{\gamma}$ consists of $r+q+s$ vectors. Furthermore, since $\{w_1, w_2, \ldots, w_q, u_1, u_2, \ldots, u_s\}$ is a basis for $E_\lambda = N(U)$, it follows that $\text{nullity}(U) = q+s$

Therefore $\dim(K_\lambda) = \text{rank}(U) + \text{nullity}(U) = r+q+s$

So, $\bar{\gamma}$ is a linearly independent subset of K_λ containing $\dim(K_\lambda)$ vectors. It follows that $\bar{\gamma}$ is a basis for K_λ

<u>Corollary 7.1.13</u> Let T be a linear operator on a finite dimensional vector space V whose characteristic polynomial splits. Then T has a Jordan Canonical form.

Proof: Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. By Theorem 7.1.12, for each i, there is an ordered basis β_i consisting of a disjoint union of cycles of generalized eigenvectors corresponding to λ_i. Let $\beta = \beta_1 \cup \beta_2 \cup \dots \cup \beta_k$. Then, by Theorem 7.1.6 (b) (page-142), β is an ordered basis for V.

The Jordan Canonical form also can be studied from the viewpoint of matrices

Definition 7.1.14 Let $A \in M_{n\times n}(F)$ be such that the characteristic polynomial of A (and hence of L_A) splits. Then the Jordan Canonical form of A is defined to be the Jordan Canonical form of the linear operator L_A on F^n

The next result is an immediate consequence of this definition and corollary 7.1.13

Corollary 7.1.15 Let A be an $n\times n$ matrix whose characteristic polynomial splits. Then A has a Jordan canonical form J, and A is similar to J.

Proof: Exercise.

We can now compute the Jordan Canonical form of matrices and linear operators in some simple cases, as is illustrated in the next two examples. The ~~tool~~ tools necessary for computing the Jordan Canonical forms in general are developed in the next section.

Example 2

Let $A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix} \in M_{3\times 3}(\mathbb{R})$

To find the Jordan Canonical form for A, we need to find a Jordan Canonical basis for $T = L_A$

The characteristic polynomial of A is

$$f(t) = \det(A - tI) = -(t-3)(t-2)^2.$$

Hence $\lambda_1 = 3$ and $\lambda_2 = 2$ are the eigenvalues of A with multiplicities 1 and 2, respectively. By Theorem 7.1.6, $\dim(K_{\lambda_1}) = 1$ and $\dim(K_{\lambda_2}) = 2$. By Theorem 7.1.4, $K_{\lambda_1} = N(T-3I)$, and $K_{\lambda_2} = N((T-2I)^2)$. Since $E_{\lambda_1} = N(T-3I)$, we have that $E_{\lambda_1} = K_{\lambda_1}$. Observe that $(-1,2,1)$ is an eigenvector of T corresponding to $\lambda_1 = 3$; therefore $\beta_1 = \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$ is a basis for K_{λ_1}.

Since $\dim(K_{\lambda_2}) = 2$ and a generalized eigenspace has a basis consisting of a union of cycles, this basis is either a union of two cycles of length 1 or single cycle of length 2. The former case is impossible, because the vectors in the basis would be eigenvectors - contradicting the fact that $\dim(E_{\lambda_2}) = 1$. Therefore the desired basis is a single cycle of length 2. ~~The former case is impossible because the vectors in the basis would be eigenvectors~~ A vector v is the end vector of such a cycle if and only if $(A-2I)v \neq \theta$, but $(A-2I)^2(v) = \theta$. It can be easily shown that

$\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$ is a basis for the solution space of the homogeneous system $(A-2I)^2 x = \theta$. Now choose a vector v in this set so that $(A-2I)(v) \neq \theta$. The vector $v = (-1, 2, 0)$ is an acceptable candidate for v. Since $(A-2I)(v) = (1, -3, -1)$, we obtain the cycle of generalized eigenvectors.

$$\beta_2 = \{(A-2I)v, v\} = \left\{ \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$$ as a basis for K_{λ_2}.

Finally, we take the union of these two bases to obtain

$$\beta = \beta_1 \cup \beta_2 = \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\},$$

which is a Jordan canonical basis for A. Therefore,

$$J = [T]_\beta = \left(\begin{array}{c|cc} 3 & 0 & 0 \\ \hline 0 & 2 & 1 \\ 0 & 0 & 2 \end{array} \right)$$ is a Jordan canonical form of A. Notice that A is similar to J. In fact, $J = Q^{-1} A Q$, where Q is the matrix whose columns are the vectors in β.

Example 3 Let T be the linear operator on $P_2(\mathbb{R})$ defined by $T(g(x)) = -g(x) - g'(x)$. We find a Jordan canonical form of T and a Jordan canonical basis for T.

Let β the standard ordered basis for $P_2(\mathbb{R})$. Then

$$[T]_\beta = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$ which has the characteristic polynomial $= -1(t+1)^3$. Thus

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