So, $Ax_3 = b$ and $x_3 \geq 0$. So, $x_3 \in X$.

Hence X is a convex set.

Theorem 3.3.2 All BFS of the convex set all feasible solutions $X = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ are extreme points of X and conversely.

Proof: Here the rank of the matrix $A = [a_{ij}]_{m \times n}$ is m.

Let $a_1, a_2, \ldots, a_n$ be the columns of A and without loss of generality, assume that $B = (a_1, a_2, \ldots, a_m)$ be a basis matrix where $a_1, a_2, \ldots, a_m$ are the column vectors corresponding to the first m variables $x_1, x_2, \ldots, x_m$. Let x be the corresponding BFS and x is given by $x = [x_B, 0]$ (writing the column vectors in third bracket) . . . (1)

where $x_B = B^{-1}b$ is an m component column vector and 0 is a $(n-m)$ component null column vector.

We have to prove that x is an extreme point of X.

If possible, let x be not an extreme point of the convex set X. So, $\exists$ two points $x_1 \neq x_2$ in X such that $x = \lambda x_1 + (1-\lambda) x_2$, $0 < \lambda < 1$

where $x_1 = [u_1, v_1]$ and $x_2 = [u_2, v_2]$ . . . (2)

where u_1 contains m components of x_1, corresponding to the variables $x_1, x_2, \ldots, x_m$ and v_1 contains the remaining $(n-m)$ components of x_1. Similarly u_2 and v_2 are constructed.

So, $x = \lambda [u_1, v_1] + (1-\lambda) [u_1, v_1]$

$= [\lambda u_1 + (1-\lambda) u_2,\ \lambda v_1 + (1-\lambda) v_2]$

As $x = [x_B, 0]$, then equating the component corresponding to the last $(n-m)$ variables, we get

$\lambda v_1 + (1-\lambda) v_2 = 0$ which is possible only if when $v_1 = 0$ and $v_2 = 0$ as $v_1 \geq 0$, $v_2 \geq 0$ and $0 < \lambda < 1$.

So, $x_1 = [u_1, 0]$, $x_2 = [u_2, 0]$

Hence u_1 and u_2 are the m components of the solution set corresponding to the basic variables $x_1, x_2, \ldots, x_m$ for which the basis matrix is B.

So, $u_1 = B^{-1}b$ and $u_2 = B^{-1}b$. Hence $x_B = u_1 = u_2$

So, three points x, x_1, x_2 are not different and so, x can not be expressed as the convex combination of two distinct points. So, a BFS x is an extreme point. So, all BFS are extreme points of X.

Conversely, let x be an extreme point of the convex set X. So, $Ax = b$ and $x \geq 0$

we have to prove that x is a BFS.

Let $x = [x_1, x_2, \ldots, x_k, \overbrace{0, 0, \ldots, 0}^{n-k}]$ where $x_j > 0$, $j = 1, 2, \ldots k$. If the column vectors $a_1, a_2, \ldots, a_k$ associated with the variables $x_1, x_2, \ldots, x_k$ respectively are linearly independent (which is possible only for $k \leq m$) then x, the extreme point of the convex set X is also a BFS and we have nothing to prove. If $a_1, a_2, \ldots, a_k$ are not linear independent, then

$$\sum_{j=1}^{k} x_j a_j = b \quad \cdots \cdots (3)$$

and $\exists$ $\lambda_1, \lambda_2, \ldots \lambda_k$ such that

$$\sum_{j=1}^{k} \lambda_j a_j' = 0 \quad \cdots \cdots (4)$$

with at least one $\lambda_j \neq 0$

Let $\delta > 0$, then from (2) and (3), we get

$$\sum_{j=1}^{k} (x_j \pm \delta \lambda_j) a_j' = b \quad \cdots \cdots (5)$$

Consider δ in the interval $0 < \delta < l$

where $l = \min\limits_{j} \left(\frac{x_j'}{|\lambda_j|}, \lambda_j \neq 0 \right)$

Then $x_j \pm \delta \lambda_j \geq 0$ for $j = 1, 2, \ldots, k$

Hence the two points,

$$x_1 = [x_1 + \delta\lambda_1, x_2 + \delta\lambda_2, \ldots, x_k + \delta\lambda_k, \overbrace{0, 0, \ldots 0}^{n-k}]$$

and $x_2 = [x_1 - \delta\lambda_1, x_2 - \delta\lambda_2, \ldots, x_k - \delta\lambda_k, \overbrace{0, 0, \ldots 0}^{n-k}]$

are two points of the convex set X.

Now $x = \frac{1}{2}x_1 + \frac{1}{2}x_2$. So, x can be expressed as

$$x = \lambda x_1 + (1-\lambda) x_2 \quad \text{where } \lambda = 1/2$$

So, x is not an extreme point, contradicting our assumption that x is an extreme point. So, the column vectors $a_1, a_2, \ldots a_k$ are linearly independent, and hence x is a B.F.S.

<u>Theorem 3.3.3</u> If a LPP admits an optimal solution, then the objective function has its optimal value at an extreme point of the convex polyhedron generated by the set of feasible solution

(Here we assume the result that a closed convex set which is bounded from below has extreme points in every supporting hyperplane)

Proof: Let X be the convex polyhedron generated by the set of feasible solutions of an LPP. Let cx be the objective function and the LPP is a maximization problem. Assume that maximum value of cx is z_0 corresponding to a point or points of X. We are to prove that $\exists$ at least one extreme point of X which makes the objective function a maximum. Here $cx = z_0$ is the optimal hyperplane. Hence it is a supporting hyperplane of X. Let $T = X \cap \{x \in \mathbb{R}^n : cx = z_0\}$

Then $T \neq \phi$ as by assumption, there is at least one point which is both in X and in $cx = z_0$. Let x_0 be an extreme point of T. Then $cx_0 = z_0$ and there do not exist two different points in T such that x_0 can be expressed as the convex combination of these two points. Again as $x_0 \in T$, $x_0 \in X$. If possible, let x_0 be not an extreme point of X. So, $\exists$ two points x_1 and x_2 in X such that

$$x_0 = \lambda x_1 + (1-\lambda)x_2, \quad 0 < \lambda < 1 \quad \dots (1)$$

$$\text{or, } cx_0 = \lambda cx_1 + (1-\lambda)cx_2 = z_0 \quad \dots (2)$$