But $cx_1 \le z_0$ and $x_2 \le z_0$ (as z_0 is the maximum value of cx)

So, (2) will be satisfied if and only if $cx_1 = z_0$ and $cx_2 = z_0$ which indicates $x_1, x_2 \in T$ and it contradicts that x_0 is an extreme point of T. So, x_0 is also an extreme point of X. which makes cx maximum. Finally, we are to show that T has at least one extreme point.

~~T is the intersection of two closed convex sets, one $CX = z_0$ and the other X of H_0 which is also atleast bounded from below. Hence~~

Here X is a closed convex set which is bounded from below has at least one extreme point on the optimal hyperplane $cx = z_0$ (which is a supporting hyperplane of X)

So, T has at least one extreme point.

Theorem 3.3.4 (Fundamental Theorem of Linear Programming)

If an LPP, optimize $z = cx$ subject to $Ax = b$, $x \geq 0$ where $A = [a_{ij}]_{m\times n}$ is the coefficient matrix $(m<n)$ and $r(A) = m$ (where $r(A)$ = rank of A), $x = [x_1, x_2, \ldots, x_n]$ $b = [b_1, b_2, \ldots, b_m]$, has an optimal solution then $\exists$ at least one BFS which will be optimal.

Proof: Case I Problem of maximization

Let $x_0 = [x_1, x_2, \ldots, x_n]$ be an optimal solution of the problem which makes the objective function maximum. Without loss of generality, we assume that first k components are positive and the last $(n-k)$ components are zero. So, the optimal solution $x_0 = [x_1, x_2, \ldots, x_k, \overbrace{0, 0, \ldots 0}^{n-k}]$ and $z_0 = cx_0 = \sum_{j=1}^{k} c_j x_j$

If $a_1, a_2, \ldots, a_k$ be the column vectors associated with the variables $x_1, x_2, \ldots, x_k$ then the optimal solution will be a BFS provided that the vectors $a_1, a_2, \ldots, a_k$ are linearly independent which is possible only for $k \le m$. If $a_1, a_2, \ldots, a_k$ are not linearly ~~dep~~ independent, then the solution is not a BFS. Now $\sum_{j=1}^{k} x_j a_j = b$, $x_j > 0$, $j = 1, 2, \ldots, k$ $\quad \ldots (1)$

as as $a_1, a_2, \ldots, a_k$ are not linearly independent there exist $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\sum_{j=1}^{k} \lambda_j a_j = 0$ $\quad \ldots (2)$

with at least one $\lambda_j > 0$.

Taking $v = \max_j \left\{ \frac{\lambda_j}{x_j} \right\}$ which is a positive quantity, the solution set

$$x_0' = [x_1', x_2', \ldots, x_k'; 0, 0, \ldots, 0]$$

where $x_j' = x_j - \frac{\lambda_j}{v} \geq 0$ $[j = 1, 2, \ldots, k]$, is also a feasible solution which contains maximum $(k-1)$ positive variables. Let ~~there~~ without loss of generality $x_k' = 0$, then

$$x_0' = [x_1', x_2', \ldots, x_{k-1}', \overbrace{0, 0, \ldots, 0}^{n-k+1}]$$

The value of the objective function for the solution set x_0' is

$$z_0' = \sum_{j=1}^{k-1} c_j x_j' = \sum_{j=1}^{k} c_j x_j' \quad [\text{as } x_k' = 0]$$

$$= \sum_{j=1}^{k} c_j \left(x_j - \frac{\lambda_j}{v} \right) = \sum_{j=1}^{k} c_j x_j - \frac{1}{v} \sum_{j=1}^{k} c_j \lambda_j$$

$$= z_0 - \frac{1}{v} \sum_{j=1}^{k} c_j \lambda_j \quad \ldots (3)$$

If $\sum_{j=1}^{k} c_j \lambda_j = 0$, then $z_0' = z_0$ and the solution set x_0' is also an optimal solution. If the column vectors corresponding to the variables $x_1', x_2', \ldots, x_{k-1}'$

~~than the solution is a BFS which is also an optimal~~
~~solutions~~ are linearly independent then the solution is a BFS, which is also an optimal solution. If the column vectors are not linearly independent then repeating the process of reducing the number of non-zero variables for a finite number of times, ultimately an optimal solution is obtained which is a BFS and then the theorem is proved.

Now we prove that $\sum_{j=1}^{k} c_j \lambda_j = 0$

If $\sum_{j=1}^{k} c_j \lambda_j \neq 0$ then $\sum_{j=1}^{k} c_j \lambda_j > 0$ or < 0

Multiply $\sum_{j=1}^{k} c_j \lambda_j$ by a quantity $\delta (\neq 0)$ such that

$$\delta \sum_{j=1}^{k} c_j \lambda_j > 0 \quad \cdots (4)$$

Hence $\sum_{j=1}^{k} c_j x_j + \delta \sum_{j=1}^{k} c_j \lambda_j > \sum_{j=1}^{k} c_j x_j$.

$$\text{or, } \sum_{j=1}^{k} c_j (x_j + \delta \lambda_j) > z_0 \quad \cdots (5)$$

Multiplying (2) by δ, adding it with (1) we get

$$\sum_{j=1}^{k} (x_j + \delta \lambda_j) a_j = b \quad \cdots (6)$$

which indicates that $x_j + \delta \lambda_j$, $j = 1, 2, \ldots, k$ and $x_{k+1} = x_{k+2} \cdots = x_n = 0$ is a solution set of $Ax = b$.

If $x_j + \delta \lambda_j \geq 0$ $(j = 1, 2, \ldots, k)$ for particular values of δ (which is always possible, if we choose

$$\max_j \left(-\frac{x_j}{\lambda_j} ; \lambda_j > 0 \right) \leq \delta \leq \min_j \left(-\frac{x_j}{\lambda_j} ; \lambda_j < 0 \right))$$

then the solution set $x_j + \delta \lambda_j$ $(j = 1, 2, \ldots, k)$, $x_{k+1} = x_{k+2} = \cdots = x_n = 0$

Since a feasible solution of $Ax = b$

Now from (6), this solution contradicts the fact that x_0 is an optimal solution. So, our assumption is wrong. So, $\sum_{j=1}^{r} c_j \lambda_j = 0$.

Case 2 Problem of minimization : Proceeding similarly as in the case of maximization, the theorem can also be proved for minimization problem.

Reduction of a FS to a BFS

Theorem 3.3.5 (Reduction Theorem): If an LPP optimize, $z = cx$, $Ax = b$, $x \geq 0$ where $A = [a_{ij}]_{m \times n}$ is the coefficient matrix $(n > m)$ and $r(A) = m$, has one FS (feasible solution) then it has atleast one BFS (basic feasible solution).

Proof: Let ~~$x_0 = [x_1, x_2 \ldots]$~~ x_0 be a feasible solution to the LPP. Without loss of generality, we assume that the first p components are positive and the last $(n-p)$ components are zero, so,

$$x_0 = [x_1, x_2, \ldots, x_p, \overbrace{0, 0, 0, \ldots, 0}^{n-p}]$$

Now let $A = [a_1, a_2, \ldots a_p, \ldots a_n]$ where $a_1, a_2, \ldots a_p, \ldots, a_n$ are column vectors of A. As x_0 is a feasible solution, we have $Ax_0 = b$ and $x_0 \geq 0$

So, $$\sum_{j=1}^{p} a_j x_j = b \quad \ldots (1) \qquad \text{and } x_j > 0,\ j = 1, 2, \ldots, p \quad \ldots (2)$$

$a_1, a_2, \ldots, a_p$ are either linearly independent or linearly dependent. In the former case, x_0 will be a basic feasible solution, being non-degenerate, if $p = m$ and degenerate, if $p < m$.