In the later case, we shall try to show that we can reduce the number of positive components of x_0 step by step until the column vectors associated with the positive variables are linearly independent.

Assume in this case that the vectors $a_1, a_2, \ldots, a_p$ are linearly independent, so that

$$\sum_{j=1}^{p} \lambda_j a_j = 0 \quad \text{in which} \quad \ldots (3)$$

at least one of the λ_j's is non-zero

Let $\lambda_r \neq 0$; then we have

$$a_r = -\sum_{\substack{j=1 \\ j\neq r}}^{p} \frac{\lambda_j}{\lambda_r} a_r \text{ and } j \neq r$$

Putting this in (1), we have

$$\sum_{\substack{j=1 \\ j\neq r}}^{p} \left(x_j - x_r \frac{\lambda_j}{\lambda_r} \right) a_j = b \quad \ldots (4)$$

This shows that,

$\left(x_j - x_r \frac{\lambda_j}{\lambda_r} \right)$, for $j = 1, 2, \ldots (r-1), (r+1), \ldots, p$ together with $(n-p+1)$ zeroes is a solution to $Ax = b$. Let this solution be x_0'. So, x_0' is a solution of $Ax = b$ with not more than $p-1$ non-zero variables as $x_r - x_r \frac{\lambda_r}{\lambda_r} = 0$.

Now we choose properly a_r so that these $(p-1)$ non-zero components of x_1 be non-negative. Then x_1 will be a feasible solution. For this, we have $x_j - x_r \frac{\lambda_j}{\lambda_r} \geq 0 \quad \ldots (5)$

or,
$$\left.\begin{aligned} \frac{x_j}{\lambda_j} &\geq \frac{x_r}{\lambda_r}, \text{ if } \lambda_j > 0 \\ \text{and } \frac{x_j}{\lambda_j} &\leq \frac{x_r}{\lambda_r} \text{ if } \lambda_j < 0 \end{aligned}\right\} \quad \text{---}(6)$$

For any j, for which $\lambda_j = 0$, (5) will be automatically satisfied. So, we choose ~~$\frac{x_r}{\lambda_r}$~~ ~~such that~~ as

~~$\text{Max}\left\{\frac{x_j}{\lambda_j} : \lambda_j > 0\right\} \leq \frac{x_r}{\lambda_r}$~~

such that (6) holds. i.e, we choose $\frac{x_r}{\lambda_r}$ such that

$$\text{Max}\left\{\frac{x_j}{\lambda_j} : \lambda_j > 0\right\} \leq \frac{x_r}{\lambda_r} \leq \text{Min}\left\{\frac{x_j}{\lambda_j} : \lambda_j < 0\right\} \quad \text{---}(7)$$

This value of $\frac{x_r}{\lambda_r}$ will satisfy (6) and hence the new solution x_1 will be a feasible solution.

So, we have constructed a solution with not more ~~that~~ than $(p-1)$ positive variables, the remaining variables being zero. If now the vectors associated with the positive variables in the ~~so~~ solution x_1 be linearly independent, then this is a basic feasible solution. If again this is not a basic feasible solution, then we can repeat the same procedure and ultimately arrive at a solution in which the vectors associated with the positive variables are linearly independent and that will give us a basic feasible solution.

4. Standard form of an LPP and Simplex method

If a constraint in an LPP, is '$\leq$' type i.e if it is

$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$, then we can add a non-negative variable x_{n+1} to the left hand side and make it as a equality as

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + x_{n+1} \leq b_i$$

Here $x_{n+1} \geq 0$; This variable is called a slack variable. Similarly, for a '$\geq$' type inequality constraint

$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n \geq b_j$ can be made a equality by adding a non-negative variable x_{n+2} which is $a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n + x_{n+2} = b_j$

Here $x_{n+2} \geq 0$; This variable is called a ~~slack~~ surplus variable. So, after the introduction of slack and surplus variables, we can write the constraints of an LPP as $Ax = b$, $x \geq 0$

where $A = [a_{ij}]_{m \times n}$ $x = [x_1, x_2, \ldots x_n]$ $b = [b_1, b_2, \ldots b_m]$

(column vectors are written in Third bracket)

So, the standard form of an LPP is

Optimize (Maximize or Minimize) $Z = cx$

subject to $Ax = b$, $x \geq 0$

In an LPP, any maximization problem can be transformed to minimization problem and vice versa as

$$\text{Max } Z = -\text{Min}(-Z)$$

and $$\text{Min } Z = -\text{Max}(-Z)$$

Henceforth, we will consider the standard form of an LPP as a maximization problem and will be written as

$$\text{Maximize } Z = cx$$
$$\text{subject } Ax = b$$
$$x \geq 0$$

Note: We take 0 as cost coefficients for slack & surplus variables. So, the values of slack and surplus variables do not affect the objective function.

4.1 Some Notations

After introducing the slack and surplus variables, let the linear programming problem be as below:

$$\text{Maximize } Z = cx$$
$$\text{subject } Ax = b$$

Where $A = [a_{ij}]_{m \times n} = [a_1, a_2, \ldots, a_n]$ where $a_1, a_2, \ldots, a_n$ are columns of A, $x = [x_1, x_2, \ldots, x_n]$, $b = [b_1, b_2, \ldots, b_m]$ and $c = (c_1, c_2, \ldots, c_n)$. [$x$ and b are column vectors and c a row vector]

Also, $r(A) = m < n$

Let B be a $m \times m$ basis matrix of A which is non-singular. Let $B = [\beta_1, \beta_2, \ldots, \beta_m]$ where $\beta_1, \beta_2, \ldots, \beta_m$ are some of the m columns of A, i.e., each β_i, $i = 1, 2, \ldots, m$ is some a_j, $j = 1, 2, \ldots, n$.

As B is a basis matrix, $\{\beta_1, \beta_2, \ldots, \beta_m\}$ forms a basis of $\mathbb{R}^m$. Let the basic variables corresponding to the vectors $\beta_1, \beta_2, \ldots, \beta_m$ of B be denoted by

$$x_B = [x_{B_1}, x_{B_2}, \ldots, x_{B_m}]$$

A basic feasible solution will contain zeros besides the basic variables x_B. So, the system $Ax = b$ reduces to $Bx_B = b$. So, we get the basic variables as $x_B = B^{-1}b$.