As $\{\beta_1, \beta_2, \dots, \beta_m\}$ is a basis of $\mathbb{R}^m$, any column a_j of A may be expressed as a linear combination of $\{\beta_1, \beta_2, \dots, \beta_m\}$

Let $a_j = y_{1j}\beta_1 + y_{2j}\beta_2 + \dots + y_{mj}\beta_m$

$$= \sum_{i=1}^{m} y_{ij}\beta_i = B y_j$$

where $y_j = [y_{1j}, y_{2j}, \dots, y_{mj}]$ $\quad j = 1, 2, \dots, n$

$$\text{So, } y_j = B^{-1}a_j \quad \dots (1) \quad j = 1, 2, \dots, n$$

Corresponding to basic variables x_B, let the prices be denoted by c_B (a row vector) as

$$c_B = (c_{B_1}, c_{B_2}, \dots, c_{B_m})$$

i.e., c_{B_i} is the price of the variable x_{B_i}, $i = 1, 2, \dots, m$

For any basic feasible solution, the value of the objective function is given by $Z_B = c_B x_B$ as all other variables are zero.

For our future reference, we introduce a new quantity z_j defined by

$$z_j = c_B y_j = y_{1j}c_{B_1} + y_{2j}c_{B_2} + \dots + y_{mj}c_{B_m}, \quad j = 1, 2, \dots, n$$

$z_j - c_j$ is called as net evaluation, $j = 1, 2, \dots, n$

Consider the following LPP : Maximize $z = 2x_1 - 4x_2$

Subject to $2x_1 + 5x_2 \geq 12$

$3x_1 + 8x_2 \leq 20$

$x_1, x_2 \geq 0$

Introducing ~~slack~~ surplus variable x_3 and slack variable x_4, the LPP can be written in the standard form

$$\text{Maximize } z = 2x_1 - 4x_2 + 0.x_3 + 0.x_4$$

subject to

$$2x_1 + 5x_2 - x_3 = 12$$

$$3x_1 + 8x_2 + x_4 = 20$$

or, Maximize $z = cx$

subject to $Ax = b$

$x \geq 0$

where $A = \begin{bmatrix} 2 & 5 & -1 & 0 \\ 3 & 8 & 0 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 12 \\ 20 \end{bmatrix}$

$r(A) = 2$ Here $a_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $a_2 = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$, $a_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $a_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

we choose our initial basis matrix B using column a_1 and a_4

So, $B = [\beta_1, \beta_2] = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$

So, $B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$

Hence the basic feasible solution is given by

$$x_B = B^{-1}b = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 12 \\ 20 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} x_{B_1} \\ x_{B_2} \end{bmatrix}$$

the remaining non basic variables $x_2 = x_3 = 0$

In the objective function $c_B = (2, 0)$ as

c_{B_1} = coefficient of x_{B_1} = coefficient of $x_1 = 2$

c_{B_2} = coefficient of x_{B_2} = coefficient of $x_4 = 0$

z_B = the value of the objective function $= c_B x_B$

$$= [2, 0]\begin{bmatrix} 6 \\ 2 \end{bmatrix} = 12$$

Any vector a_j can be expressed as a linear combination of the vectors β_1, β_2

As, $a_1 = \beta_1$ and $a_4 = \beta_2$

So, $y_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $y_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Also $y_2 = B^{-1} a_2 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix}$

Similarly, $y_3 = B^{-1} a_3 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix} = \begin{bmatrix} y_{13} \\ y_{23} \end{bmatrix}$

$z_1 = C_B y_1 = 2$, $z_2 = C_B y_2 = 5$, $z_3 = C_B y_3 = -1$

and $z_4 = C_B y_4 = 0$

Note: For any basic column a_j, $z_j = c_j$ as

$y_j = [0, 0, \cdots \overset{j\text{th component}}{1}, 0, \cdots 0]$ and $C_B y_j = c_j$ or $z_j = c_j$

4.2 Simplex method: It is an iterative process to solve a linear programming problem. In this method, we first take a BFS and check whether it is optimal or not. If the BFS is optimal, our solution is reached and the process is completed. If optimality is not satisfied, we improve the solution by getting a new BFS for which the value of the objective function is improved in the sense that it is greater than the previous value if it is a maximization problem or less if it is a minimization problem and again check whether it is optimal or not. In This way, after a finite number of steps we get an optimal solution as confirmed by fundamental theorem of the problem has optimal solution. Otherwise, the problem has

has unbounded solution.

4.3 IMPROVEMENT OF A SOLUTION

Now let Maximize $z = cx$

Subject to. $Ax = b$

$x \geq 0$

be an LPP. Let $A = [a_{ij}]_{m \times n} = [a_1, a_2, \dots, a_n]$

$b = [b_1, b_2, \dots b_m]$ $x = [x_1, x_2, \dots x_n]$ and $r(A) = m$ $(< n)$.

Let $x_B = B^{-1} b$ be a BFS for the basis matrix $B = [\beta_1, \beta_2, \dots, \beta_m]$ and let it be not optimal. So, we have to go to a new BFS which will improve the solution.

Now a vector a_j which is not a vector of B, can be expressed as

$$a_j = \sum_{i=1}^{m} y_{ij} \beta_i \quad \cdots \quad (1)$$

Now if some $y_{rj} \neq 0$, then we know that we can replace the vector β_r from B by a_j still maintaining the basis character of B. If $y_{rj} \neq 0$ and if we replace β_r from B by a_j then we can write β_r in terms of a_r and remaining vectors of B as (from (1))

$$\beta_r = \frac{1}{y_{rj}} a_j - \sum_{\substack{i=1 \\ i \neq r}}^{m} \left(\frac{y_{ij}}{y_{rj}} \right) \beta_i \quad \cdots \quad (2)$$

Now, for the basic feasible solution $x_B = B^{-1} b$, we have $Bx_B = b$

$$\text{or, } \sum_{i=1}^{m} x_{B_i} \beta_i = b \quad \cdots \quad (3)$$