z_B, the value of the objective function for the basic feasible solution x_B, is

$$z_B = c_B x_B = \sum_{i=1}^{m} c_{Bi} x_{Bi} \quad \cdots (4)$$

Using (2) in (3), we have

$$\sum_{\substack{i=1 \\ i\neq r}}^{m} x_{B_i} \beta_i + x_{Br}\left(\frac{1}{y_{rj}} a_j - \sum_{\substack{i=1 \\ i\neq r}}^{m} \left(\frac{y_{ij}}{y_{rj}}\right) \beta_i\right) = b$$

or, $$\sum_{\substack{i=1 \\ i\neq r}}^{m} \left(x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{Br}\right)\beta_i + \frac{x_{Br}}{y_{rj}} a_j = b \quad \cdots (5)$$

So, a new basic solution of the problem is

$x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{Br}$, $i = 1, 2, \ldots r-1, r+1, \ldots, m$ together with $\frac{x_{Br}}{y_{rj}}$ and the remaining $(n-m)$ components zeros. $\Big\} \cdots (6)$

In order that this solution will be feasible in addition to being basic, we must have

$$x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{Br} \geq 0, \; i \neq r \quad \cdots (7)$$

and $$\frac{x_{Br}}{y_{rj}} \geq 0 \quad \cdots (8)$$

If the conditions of feasibility are satisfied and this new solution gives an improved value of the objective function than that given by x_B, then we shall desire this solution and discard x_B.

To realise this, we have two arbitrary quantities to select and they are suffixes r in β_r and j in a_j which are upto this point arbitrary, except that $y_{rj} \neq 0$

If $x_{Br} = 0$ then conditions (7) and (8) are automatically satisfied, and the solution is feasible.

If $x_{B_r} \neq 0$, i.e., if $x_{B_r} > 0$ then $y_{rj} > 0$ from (8)

Let $y_{rj} > 0$, then (7) is automatically satisfied if $y_{ij} = 0$ or $y_{ij} < 0$. Thus the feasibility conditions are to be satisfied only for which $y_{ij} > 0$.

This requires

$$\frac{x_{B_i}}{y_{ij}} - \frac{x_{B_r}}{y_{rj}} \geq 0 \ , \ y_{ij} > 0$$

$$\text{or,} \quad \frac{x_{B_r}}{y_{rj}} \leq \frac{x_{B_i}}{y_{ij}} \ , \ y_{ij} > 0 \quad \ldots (9)$$

Thus, if we choose the vector β_r such that

$$\frac{x_{B_r}}{y_{rj}} = \min_i \left\{ \frac{x_{B_i}}{y_{rj}} \ , \ y_{ij} > 0 \right\} \quad \ldots (10)$$

Then (9) is satisfied and the solution (6) is feasible too. What we need is simply to compute $\frac{x_{B_r}}{y_{rj}}$ from (10). This tells us which column β_r of the basis matrix B is to be removed, to get a better value of for z.

Now we are to choose j of a_j such that the new basic solution makes the objective function at least as great as the current basic solution. The price vector component c_{B_r} changes to c_j as β_r is changed to a_j. If z' be the new value of the objective function, then we have

$$z' = \sum_{\substack{i=1 \\ i \neq r}}^{m} c_{B_i}\left(x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{B_r}\right) + c_j \frac{x_{B_r}}{y_{rj}}$$

$$= \sum_{i=1}^{m} c_{B_i}\left(x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{B_r}\right) + c_j \frac{x_{B_r}}{y_{rj}}$$

since $c_{B_r}\left(x_{B_r} - \frac{y_{rj}}{y_{rj}} x_{B_r}\right) = 0$

$$= \sum_{i=1}^{m} c_{B_i} x_{B_i} - \frac{x_{B_r}}{y_{rj}} \sum_{i=1}^{m} c_{B_i} y_{ij} + c_j \frac{x_{B_r}}{y_{rj}}.$$

$$= z_B - \frac{x_{B_r}}{y_{rj}}\left(\sum_{i=1}^{m} c_{B_i} y_{ij} - c_j\right) \quad \text{by (4)}$$

$$= z_B - \frac{x_{B_r}}{y_{rj}}\left(z_j - c_j\right) \quad \ldots \quad (11)$$

From (11), we see that $z' > z_B$, if $z_j - c_j$ be negative for $\frac{x_{B_r}}{y_{rj}} \geq 0$, by (8). So, by choosing any vector a_j for which $z_j - c_j$ is negative, we can improve the value of the objective function by

$$- \frac{x_{B_r}}{y_{rj}}\left(z_j - c_j\right)$$

As, we are considering always the maximization problem, we select a_j for β_r such that

$\frac{x_{B_r}}{y_{rj}}\left(z_j - c_j\right)$ is minimum most negative, i.e., which makes $z_j - c_j$ minimum most negative.

So, we choose a_k as the entering vector in the basis where $z_k - c_k = \underset{j}{\mathrm{Min}}\left\{z_j - c_j : z_j - c_j < 0\right\}$ ... (12)

In this way, we improve the value of the objective function and the process is to be continued until there are no vector a_j such that $z_j - c_j < 0$.

This method is iteratively to get a basic feasible solution from another basic feasible solution with an improved value of the objective function so long an a_j is obtained with $z_j - c_j < 0$ and at least one $y_{ij} > 0$

The quantities $(z_j - c_j)$ are called net ~~evaluation~~ evaluations corresponding to a_j.

In the illustration of Page-50, we have $z_1 - c_1 = z_4 - c_4 = 0$
and $z_2 - c_2 = 5-(-4) = 9$ and $z_3 - c_3 = -1-0 = -1$
So, only $z_3 - c_3$ is negative and hence we choose a_3 which enters the basis. To determine the departing vector which leaves the basis, we choose

$$\frac{x_{Br}}{y_{rj}} = \underset{i}{Min}\left\{\frac{x_{Bi}}{y_{ij}},\ y_{ij} > 0\right\}$$

$$= \cancel{Min\left\{\frac{x_1}{y_{13}}, \frac{x_4}{y_{43}}\right\}} \ \frac{x_4}{y_{43}}$$

~~But here~~ as $y_{13} = -\frac{1}{2} < 0$ and $y_{43} = \frac{3}{2} > 0$

So, a_4 will leave the basis

Original value of the objective function was

$$z_B = c_B x_B = 12$$

According to (11), the new value of the objective function

$$z' = 12 - \frac{2}{3/2}(-1) = \frac{40}{3}$$

So, $z' > z_B$

Note: If x_B be not degenerate, $\frac{x_{Br}}{y_{rj}}$ is positive and we have a definite increase in z and $z' > z_B$.
But if x_B be degenerate and we choose the entering vector for which $z_j - c_j < 0$ and at least $y_{ij} > 0$, we then may or may not have an increase in the objective function z, depending on whether $\frac{x_{Br}}{y_{rj}} > 0$.

4.4. Optimality condition

If, for a basic feasible solution x_B of an LPP

$$\text{Maximize } z = cx$$
$$\text{Subject to } Ax = b,\ x \geq 0,$$