we have $z_j - c_j \geq 0$ for each column a_j of A, then x_B is an optimal solution

Proof:

Let $B = (\beta_1, \beta_2, \ldots, \beta_m)$ be the basis matrix corresponding to the basic feasible solution x_B so that

$$Bx_B = b$$

Let z_B be the value of the objective corresponding to x_B i.e., $z_B = c_B x_B$

If $x = [x_1, x_2, \ldots, x_n]$ be any other feasible solution of the LPP for which the value of the objective function is z, then we have

$$Bx_B = b = Ax$$

so that $x_B = B^{-1}(Ax) = (B^{-1}A)x = Yx \quad \ldots (1)$

where $B^{-1}A = [y_1, y_2, \ldots, y_n] = Y$ where $B^{-1}a_j = y_j$, $j = 1, 2, \ldots, n$

Equating the ith component of (1) from both sides, we have,

$$x_{B_i} = \sum_{j=1}^{n} y_{ij} x_j \quad \ldots (2)$$

It is given that for every column a_j of A

$$z_j - c_j \geq 0 \quad \text{or} \quad z_j \geq c_j \quad \text{for } j = 1, 2, \ldots, n \quad \ldots (3)$$

Hence, we get $z_j x_j \geq c_j x_j$ $j = 1, 2, \ldots, n$ as $x_j \geq 0$

or, $\displaystyle\sum_{j=1}^{n} z_j x_j \geq \sum c_j x_j$

or, $\displaystyle\sum_{j=1}^{n} x_j (c_B y_j) \geq z$

or, $\displaystyle\sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} c_{B_i} y_{ij}\right) \geq z$

or, $\sum_{i=1}^{m} c_{B_i}\left(\sum_{j=1}^{n} x_j y_{ij}\right) \geq z$

or, $\sum_{i=1}^{m} c_{B_i} x_{B_i} \geq z$ · from (2)

or, $z_B \geq z$

This shows that z_B is the maximum value of the objective function.

This optimality criterion holds if $z_j - c_j \geq 0$ for non-basic vectors of A as, if a_j be in the basis for matrix B, then $a_j = \beta_i$

So, $a_j = 0.\beta_1 + 0.\beta_2 + \cdots + 1.\beta_i + 0.\beta_{i+1} + \cdots + 0.\beta_m$

so that $y_j = e_i$ (a unit vector with ith component 1 and all other components zeros.)

Again, since $a_j = \beta_i$, we have $c_j = c_{B_i}$

So, $z_j - c_j = c_B y_j - c_j = c_B e_i - c_j$

$= c_{B_i} - c_j = c_{B_i} - c_{B_i} = 0$

Thus if a_j be in the basis $z_j - c_j = 0$

Hence in all cases $z_j - c_j \geq 0$, $j = 1, 2, \ldots, n$.

Note: For minimization problem, the optimality condition is $z_j - c_j \leq 0$, $j = 1, 2, \ldots, n$.

4.5 <u>Unboundedness</u>

If the objective function in a maximization problem has no finite optimal value, then it can be increased arbitrarily, then the problem is said to ~~to~~ have an unbounded ~~best~~ solution.

If at any iteration of the simplex method, we get $z_j - c_j < 0$ for at least one j and for this $y_{ij} \leq 0$ for all $i = 1, 2, \ldots, m$, then the LPP admits of an unbounded solution ~~for~~ in a maximization problem.

Proof: Let x_B be the basic feasible solution of a LPP (maximization problem) with basis matrix B at any iteration in which there exists a j, such that

$z_j - c_j < 0$ and $y_{ij} \leq 0$ for $i = 1, 2, \ldots m$

If z_B be the value of the objective function for this basic feasible solution then

$$z_B = c_B x_B \quad \ldots (1)$$

and $Bx_B = b$ implies that $\sum_{i=1}^{m} x_{Bi} \beta_i = b \quad \ldots (2)$

where $B = [\beta_1, \beta_2, \ldots, \beta_m]$

In (2) adding ~~θa_j~~ θa_j ~~where~~ and subtracting θa_j, where θ is a scalar, we get

$$\sum_{i=1}^{m} x_{Bi} \beta_i + \theta a_j - \theta a_j = b \quad \ldots (3)$$

Now, we have $a_j = \sum_{i=1}^{m} y_{ij} \beta_i$

so that $-\theta a_j = -\theta \sum_{i=1}^{m} y_{ij} \beta_i$. Putting this in (3), we get

$$\sum_{i=1}^{m} x_{Bi} \beta_i + \theta a_j - \theta \sum_{i=1}^{m} y_{ij} \beta_i = b$$

or, $\sum_{i=1}^{m} (x_{B_i} - \theta y_{ij}) \beta_i + \theta a_j = b$ ... (4)

If now θ be assumed to be positive, then we have $x_{B_i} - \theta y_{ij} \geq 0$ since $y_{ij} \leq 0$, $i = 1, 2, \ldots, m$ ... (5)

Now (4) and (5) shows that we have obtained a new set of ~~solution~~ feasible solutions in which $(m+1)$ variables

$x_{B_i} - \theta y_{ij}$, $i = 1, 2, \ldots, m$ together

with θ, can be different from zero and the remaining components are zero. This solution, in general, is feasible, but not a basic feasible solution.

If z' be the value of the objective function corresponding to this feasible solution, then

$$z' = \sum_{i=1}^{m} c_{B_i} (x_{B_i} - \theta y_{ij}) + c_j \theta$$

$$= \sum_{i=1}^{m} c_{B_i} x_{B_i} - \theta \sum_{i=1}^{m} c_{B_i} y_{ij} + c_j \theta$$

$$= z_B - \theta (z_j - c_j) \quad \ldots (6)$$

From (6), we observe that z' will always be greater than z_B as $\theta > 0$ and $z_j - c_j < 0$.

As θ increases z' increases and by making θ sufficiently large we can make z' as large as possible. Hence the problem admits of an unbounded solution.

Note: For minimization problem, we can state a similar result with $z_j - c_j > 0$ and $y_{ij} \leq 0$ for $i = 1, 2, \ldots, m$