4.6 Simplex Algorithm

Here we do the calculations for a simplex method in tabular form

Example 1 Solve the following LPP by Simplex method :

Maximize $z = x_1 - x_2 + 2x_3 + 3x_4$

subject to

$$2x_1 + x_2 + 3x_3 + 2x_4 = 11$$

$$3x_1 - 3x_2 + 5x_3 + x_4 = 17,$$

$$x_i \geq 0, \quad i = 1, 2, 3, 4$$

Solution: Step 1 Search for a basis which will produce a feasible solution

Here $A = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 3 & -3 & 5 & 1 \end{bmatrix}$ Here rank of A or $r(A) = 2$

Thus the two equations are linearly independent an consistent

Here $a_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $a_3 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, $a_4 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $b = \begin{bmatrix} 11 \\ 17 \end{bmatrix}$

$c = (c_1, c_2, c_3, c_4) = (1, -1, 2, 3)$

Two vectors are required to form a basis

First consider the square matrix

$$B = (a_1, a_2) = \begin{bmatrix} 2 & 1 \\ 3 & -3 \end{bmatrix}, \quad \det B = -9 \neq 0$$

Hence B can be considered a basis matrix

$$B^{-1} = -\frac{1}{9}\begin{bmatrix} -3 & -1 \\ -3 & 2 \end{bmatrix}$$

The basic solution corresponding to the basis matrix B is $x_B = B^{-1}b = -\frac{1}{9}\begin{bmatrix} -3 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 11 \\ 17 \end{bmatrix}$

$$= \begin{bmatrix} 50/9 \\ -1/9 \end{bmatrix}$$ Here the solution is not feasible

Thus B can not be considered as an admissible basis matrix to start with the simplex procedure.

Now consider the ~~new~~ square matrix

$$B = (a_1\ a_3) = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, \quad \det B = 1 \neq 0$$

Hence B is also a basis matrix. The corresponding basic solution is

$$x_B = B^{-1} b = \frac{1}{1}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 11 \\ 17 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

which is feasible

So, $x_B = [x_{B_1}, x_{B_2}] = [x_1, x_3] = [4, 1]$

and $B = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ can be considered as as an admissible basis matrix. to start with the simplex procedure.

Step 2 Calculation of $y_j$ and $z_j - c_j$, $j = 1,2,3,4$ corresponding to the basis matrix $B = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

we have,

$$[y_1, y_2, y_3, y_4] = B^{-1}(a_1, a_2, a_3, a_4)$$

$$= \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 & 2 \\ 3 & -3 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 14 & 0 & 7 \\ 0 & -9 & 1 & -4 \end{bmatrix}$$

Thus $y_1 = \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $y_2 = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} 14 \\ -9 \end{bmatrix}$

$y_3 = \begin{bmatrix} y_{13} \\ y_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $y_4 = \begin{bmatrix} y_{14} \\ y_{24} \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$

Here $B = (a_1, a_3) = (\beta_1, \beta_2)$ say

Thus $C_B = (C_{B_1}, C_{B_2}) = (c_1, c_3) = (1, 2)$ the cost components corresponding to the basic variables $x_1$ and $x_3$.

$Z_1 - c_1 = C_B y_1 - c_1$ ~~$= C_{B_1} y_{11} + C_{B_2} y_{21} - 1 = 1 \times 1 + 2 \times 0$~~

$= C_{B_1} y_{11} + C_{B_2} y_{21} - 1 = 1 \times 1 + 2 \times 0 - 1 = 0$

$Z_2 - c_2 = C_B y_2 - c_2 = C_{B_1} y_{12} + C_{B_2} y_{22} + 1 = 1 \times 14 + 2 \times (-9) + 1$
$= -3$

$Z_3 - c_3 = C_B y_3 - c_3 = C_{B_1} y_{13} + C_{B_2} y_{23} - 2 = 1 \times 0 + 2 \times 1 - 2 = 0$

$Z_4 - c_4 = C_B y_4 - c_4 = C_{B_1} y_{14} + C_{B_2} y_{24} - 3 = 1 \times 7 + 2 \times (-4) - 3 = -4$

Value of the objective function $= Z_0 = C_B x_B = C_{B_1} x_{B_1} + C_{B_2} x_{B_2}$
$= 1 \times 4 + 2 \times 1 = 6$

Conclusion: Here $Z_2 - c_2$ and $Z_4 - c_4$ are both negative with at least one of $y_{i2}$ and $y_{i4} > 0$. Thus the BFS $x_B = [x_{B_1}, x_{B_2}] = [x_1, x_3] = [4, 1]$ corresponding to the basis matrix $B = (a_1, a_3) = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ is not an optimal solution of the problem and we shall have to proceed further to obtain an optimal solution, if it exists.

Step 3: Search for a new vector to enter in the next basis and the vector which will leave the current basis $(a_1, a_3)$

$$\min_j \left( z_j - c_j \,;\, z_j - c_j < 0 \right) = \min \left( z_2 - c_2,\ z_4 - c_4 \right)$$
$$= \min(-3, -4) = -4$$

Then the fourth vector $a_4$ will enter in the next basis. Thus the fourth column is the key column. Again

$$\min_i \left( \frac{x_{B_i}}{y_{i4}},\ y_{i4} > 0 \right) = \min \left( \frac{4}{7} \right) = \frac{4}{7}$$

$\left[ \frac{x_{B_2}}{y_{24}} = \frac{1}{-4} \text{ is not consider as } y_{24} = -4 < 0 \right]$

which occurs for $i=1$. Then first row is the key row and $y_{14} = 7$ is the key element. Then the first vector $\beta_1 = a_1$ will leave the basis and replaced by $a_4 = \beta_1^*$ (say) to form a new basis matrix. $B = (a_4, a_3) = (\beta_1^*, \beta_2) = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$

and $y_{14} = 7$ which is the element at the point of intersection of key row (first row) and the key column (fourth column) is the key element

New $x_{B_1} = x_4 = 4/7$, $x_{B_2} = x_3 = \frac{\cancel{28}}{\cancel{7}} 1 + \frac{16}{7} = \frac{23}{7}$

New $y_1 = \begin{pmatrix} y_{11} \\ y_{21} \end{pmatrix} = \begin{pmatrix} 1/7 \\ 4/7 \end{pmatrix}$ $y_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} y_{12} \\ y_{22} \end{pmatrix}$

$y_3 = \begin{pmatrix} y_{13} \\ y_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $y_4 = \begin{pmatrix} y_{14} \\ y_{24} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$