follows :

Table 2

		C	5	2	2	0	0	Min. Ratio
B	C_B	b	a_1	a_2	a_3	a_4	a_5	
a_1	5	30	1	2	-2	1	0	
a_5	0	6	0	1	$\boxed{3}$	-1	1	$6/3=2^*$
z_j-c_j		150	0	8	-12^*	5	0	

As $z_3-c_3<0$, optimal is not satisfied. Here a_3 is the entering vector and a_5 is the departing vector and we go to the next table as follows : ($y_{23}=3$ is the key element)

Table 3

		C	5	2	2	0	0
B	C_B	b	a_1	a_2	a_3	a_4	a_5
a_1	5	34	1	$8/3$	0	$1/3$	$2/3$
a_3	2	2	0	$1/3$	1	$-1/3$	$1/3$
z_j-c_j		174	0	12	0	1	4

As $z_j-c_j \geq 0$ for all j, the optimality condition is satisfied. So, an optimal solution is $x_1=34$, $x_2=0$ (non basic variable) and $x_3=2$ and the optimal, i.e., maximum value of the objective function z is 174.

Here also the optimal solution is unique as $z_j-c_j \neq 0$ for any non-basic vector.

Note: From now on, we shall write down all the simplex tables in a compact form.

Example 2 Solve the LPP by simplex method :

Maximize $z = 4x_1+3x_2$

subject to $3x_1+x_2 \leq 15$

$3x_1+4x_2 \leq 24$ $\quad x_1 \geq 0,\ x_2 \geq 0$

Solution: Adding slacking variables x_3 and x_4 to the first constraint and second constraints respectively and adjusting the objective function, the converted problem is

$$\text{Maximize } z = 4x_1 + 3x_2 + 0.x_3 + 0.x_4$$

Subject to
$$3x_1 + x_2 + x_3 = 15$$
$$3x_1 + 4x_2 + x_4 = 24$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Simplex tables are:

	~~C~~	~~4~~	~~3~~	~~0~~	~~0~~
~~B~~	~~b~~	~~a_1~~	~~a_2~~	~~a_3~~	~~a_4~~

		C	4	3	0	0	
B	C_B	b	a_1	a_2	a_3	a_4	Min. ratio
a_3	0	15	$\boxed{3}$	1	1	0	$15/3 = 5^*$
a_4	0	24	3	4	0	1	$24/3 = 8$
$z_j - c_j$		0	-4^*	-3	0	0	
a_1	4	5	1	$1/3$	$1/3$	0	$5/\frac{1}{3} = 15$
a_4	0	9	0	$\boxed{3}$	-1	1	$9/3 = 3^*$
$z_j - c_j$		20	0	$-5/3^*$	$4/3$	0	
a_1	4	4	1	0	$4/9$	$-1/9$	
a_2	3	3	0	1	$-1/3$	$1/3$	
$z_j - c_j$		25	0	0	$7/9$	$5/9$	

a_1 is entering vector, a_3 is departing vector, and 3 (enclosed in square) is the key element

a_2 is entering vector, a_4 is departing vector and 3 is the key element

So, $z_j - c_j \geq 0$ for all j in the final table. Final basis is $B = [a_1, a_2]$, optimal value of $z = \max z = 25$ and the unique optimal solution is $x_1 = 4$, $x_2 = 3$ (as no $z_j - c_j = 0$ for non-basic vector)

Example 3

Solve the LPP

Minimize $z = -2x_1 + 3x_2$

Subject to $2x_1 - 5x_2 \le 7$

$4x_1 + x_2 \le 8$

$7x_1 + 2x_2 \le 16$, $x_1 \ge 0$, $x_2 \ge 0$

Solution: The problem is a problem of minimization. Let $z' = -z$ then $\min z = -\max(-z) = -\max z'$. Hence the problem is a problem of maximization of $z' = -z = -(-2x_1 + 3x_2) = 2x_1 - 3x_2$ and finally $\min z = -\max z'$ with the same solution set. Here $b_i \ge 0$ for all i and constraints are associated with the sign '$\le$'. Introducing three slack variables x_3, x_4, and x_5 corresponding to first, second and third constraints respectively we get the following converted equations

$$2x_1 - 5x_2 + x_3 = 7$$
$$4x_1 + x_2 + x_4 = 8$$
$$7x_1 + 2x_2 + x_5 = 16, \quad x_1, x_2, x_3, x_4, x_5 \ge 0$$

The adjusted objective function is $z' = 2x_1 - 3x_2 + 0 \cdot x_3 + 0 \cdot x_4 + 0 \cdot x_5$

The simplex tables are:

		c	2	−3	0	0	0	
B	C_B	b	a_1	a_2	a_3	a_4	a_5	Min. ratio
a_3	0	7	2	−5	1	0	0	7/2
a_4	0	8	$\boxed{4}$	1	0	1	0	8/4 = 2*
a_5	0	16	7	2	0	0	1	16/7
$z_j - c_j$		0	−2*	3	0	0	0	
a_3	0	3	0	−1/2	1	−1/2	0	
a_1	2	2	1	1/4	0	1/4	0	
a_5	0	2	0	1/4	0	−7/4	1	
$z_j - c_j$		4	0	7/2	0	1/2	0	

a_1 is the entering vector, a_4 is the departing vector and 4 is the key element.

As $z_j - c_j \ge 0$, the optimality condition is satisfied. Hence $\max z' = 4$. So, $\min z = -\max z' = -4$. Hence the minimum value of z is -4 corresponding to the unique optimal solution.

$x_1 = 2$, $x_2 = 0$ (x_2 is a non-basic variable) as none of $z_j - c_j = 0$ for a non-basic vector.

Problem having Multiple optimal solutions

Example 4 Use simplex method to solve the following LPP:

$$\text{Maximize } z = 5x_1 + 2x_2$$

$$\text{Subject to } \quad 6x_1 + 10x_2 \le 30$$
$$10x_1 + 4x_2 \le 20, \quad x_1, x_2 \geq 0$$

Is the optimal solution unique? If not write down two other optimal solutions.

Solution: Adding slack variables x_3 and x_4 in the two constraints we get the two converted equations

$$6x_1 + 10x_2 + x_3 = 30$$
$$10x_1 + 4x_2 + x_4 = 20, \quad x_1, x_2, x_3, x_4 \geq 0$$

The adjusted objective function $z = 5x_1 + 2x_2 + 0.x_3 + 0.x_4$

Simplex tables are:

	c	5	2	0	0	
B	C_B	b	a_1	a_2	a_3	a_4	Min. ratio
a_3	0	30	6	10	1	0	$30/6 = 5$
a_4	0	20	$\boxed{10}$	4	0	1	$20/10 = 2^*$
$z_j - c_j$		0	-5^*	-2	0	0	
a_3	0	18	0	$\boxed{38/5}$	1	$-3/5$	$\frac{18}{38/5} = \frac{90}{38}^*$
a_1	5	2	1	$2/5$	0	$1/10$	$\frac{2}{2/5} = 5$
$z_j - c_j$		10	0	0^*	0	$1/2$	
a_2	2	$45/19$	0	1	$5/38$	$-3/38$	
a_1	5	$20/19$	1	0	$-2/19$	$5/38$	
$z_j - c_j$		10	0	0	0	$1/2$	

a_1 is the entering vector, a_4 is the departing vector and 10 is the key element. In the 2nd table, $z_j - c_j \geq 0$ for all j, so, the optimality is reached and optimal value of z = max z = 10 and an optimal solution is $x_1 = 5$, $x_2 = 0$ (x_2 is the non-basic variable)

Here $z_2 - c_2 = 0$ for the non-basic vector a_2. So the problem has multiple optimal solution. To get another optimal solution, we take a_2 as the entering vector. Then the departing vector

Here $z_j - c_j \geq 0$ for all j

So another optimal solution is $x_1 = 45/19$, $x_2 = 20/19$