		c_j	-4	-3	0	0	-M	-M	
B	c_B	b	a_1	a_2	a_3	a_4	a_5	a_6	Min. ratio
a_5	-M	8	1	$\boxed{2}$	-1	0	1	0	$8/2 = 4^*$
a_6	-M	12	3	2	0	-1	0	1	$12/2 = 6$
$z_j - c_j$		-20M	-4M+4	$-4M+3^*$	M	M	0	0	
a_2	-3	4	1/2	1	-1/2	0		0	$\frac{4}{1/2} = 8$
a_6	-M	4	$\boxed{2}$	0	1	-1		1	$\frac{4}{2} = 2^*$
$z_j - c_j$		-4M -12	$-2M + 5/2^*$	0	-M + 3/2	M		0	
a_2	-3	3	0	1	-3/4	1/4			
a_1	-4	2	1	0	1/2	-1/2			
$z_j - c_j$		-17	0	0	1/4	5/4			

a_2 is the incoming vector, a_5 is the outgoing vector and 2 is the key element. We shall leave the column a_5 for ever.

a_1 is the incoming vector, a_6 is the out going vector, and 2 is the key element. We shall leave the column a_6 for ever.

In the last table $z_j - c_j \geq 0$ for $j = 1, 2, 3, 4$. So, the solution obtained is an optimal solution. No artificial vector is present in the final basis. So, all the artificial variables are at zero level at final stage. Hence an optimal solution to the problem is $x_1 = 2$ and $x_2 = 3$ and $\max z' = -17$

So, $\min z = -\max z' = -(-17) = 17$.

Example 2 Solve the following LPP by Big M method and prove that the problem has no feasible solution.

Maximize $z = 5x_1 + 11x_2$

subject to
$$2x_1 + x_2 \leq 4$$
$$3x_1 + 4x_2 \geq 24$$
$$2x_1 - 3x_2 \geq 6, \quad x_1, x_2 \geq 0$$

Solution: After the introduction of slack, surplus and artificial variables so that there exists a unit matrix which will form an initial unit basis, the constraints are

$$2x_1 + x_2 + x_3 = 4$$
$$3x_1 + 4x_2 - x_4 + x_6 = 24$$
$$2x_1 - 3x_2 - x_5 + x_7 = 6$$

$x_1, x_2 \geq 0$, $x_3 \geq 0$ is slack variable, $x_4, x_5 \geq 0$ are surplus variables and $x_6, x_7 \geq 0$ are artificial variables and the adjusted objective function is

$$Z = 5x_1 + 11x_2 + 0.x_3 + 0.x_4 + 0.x_5 - Mx_6 - Mx_7$$ (M is a large positive number)

Simplex tables are:

		c	5	11	0	0	0	$-M$	$-M$	
B	c_B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	min. ratio
a_3	0	4	$\boxed{2}$	1	1	0	0	0	0	$4/2 = 2^*$
a_6	$-M$	24	3	4	0	-1	0	1	0	$24/3 = 8$
a_7	$-M$	6	2	-3	0	0	-1	0	1	$6/2 = 3$
$z_j - c_j$		$-30M$	$-5M-5$	$-M-11$	0	M	M	0	0	
a_1	5	2	1	$1/2$	$1/2$	0	0	0	0	
a_6	$-M$	18	0	$5/2$	$-3/2$	-1	0	1	0	
a_7	$-M$	2	0	-4	-1	0	-1	0	1	
$z_j - c_j$		$-20M+10$	0	$\frac{3}{2}M - \frac{17}{2}$	$\frac{5}{2}M + \frac{5}{2}$	M	M	0	0	

a_1 is the entering vector, a_3 is the departing vector and 2 is the key element

So, all $z_j - c_j \geq 0$ in the 2nd table. But the artificial variables are present in the basis at positive level. The only conclusion is that the problem has no feasible solution.

4.9 Two phase method

Two phase method is also a simplex method and applicable as an alternative to Charnes Big M method. That is, it is only applicable when we require to introduce at least one artificial variable to the initial unit basis. Solved problems given below will give clear idea about this method.

In this method, the problems are to be solved in two phases. In the first phase, all artificial vectors are to be removed from the basis matrix which is initially a unit basis. This can be done mathematically by the following device.

Let us consider a new objective function $U = \Sigma x_\alpha$ [x_α is an artificial variable]. From the mathematical point of view all artificial variables must be at zero level at the optimal stage. Hence the minimum value of U should be equal to zero. But practically this is not true in all problems. To get the minimum value of U, an auxiliary LPP

Minimize $U = \Sigma x_\alpha$ subject to all constraints is to be solved first and in an attempt to solve the problem following three cases may arise:

(i) Min $U = 0$ and all artificial vectors are removed from the basis. Then removing all columns corresponding to the artificial vectors from the matrix formed with the elements y_{ij}, proceed further to solve the original problem by usual method (with changed value of c and C_B) in the second phase.

(ii) Min $U = 0$, but some artificial variables are present in the optimal stage at 0 level. In that case some constraints may be redundant. Taking the final simplex table of the auxiliary LPP as the initial simplex table of the original problem (with the changed value of c and C_B) we shall have to proceed further to get the optimal value of the objective function z, of the original problem. But here the costs of artificial variables, only which are basic, are taken to be zero and drop all y_j corresponding to the non-basic artificial vectors.

(iii) Min $U > 0$ and the optimality condition are satisfied in the ~~next~~ auxiliary LPP. In that case the problem has no feasible solution and there is no question of proceeding further in the second phase

Example 1 Solve the following LPP by two phase method

Minimize $z = 3x_1 + 5x_2$

subject to
$$x_1 + 2x_2 \geq 8$$
$$3x_1 + 2x_2 \geq 12$$
$$5x_1 + 6x_2 \leq 60, \quad x_1, x_2 \geq 0$$

Solution: After the introduction of surplus variables, slack variable and artificial variables so that there exists a unit matrix which will form an initial unit basis, the constraints are

$$\begin{aligned} x_1 + 2x_2 - x_3 \quad\quad\quad + x_6 \quad &= 8 \\ 3x_1 + 2x_2 \quad - x_4 \quad\quad\quad + x_7 &= 12 \\ 5x_1 + 6x_2 \quad\quad + x_5 \quad\quad\quad &= 60, \end{aligned}$$

$x_1, x_2 \geq 0$, $x_3, x_4 \geq 0$ are surplus variables, $x_5 \geq 0$ is slack variable and $x_6, x_7 \geq 0$ are artificial variables

In the first phase, let the objective function be $U = x_6 + x_7$ of the auxiliary LPP and U is to be