We know ~~Minimi~~ $\text{Min } W = -\text{Max}(-W) = -\text{Max } W'$

Where $W' = -W = -b^t w$

So, (2) can be written as

$$\text{Maximize } W' = -b^t w$$
$$\text{Subject to } -A^t w \leq -c^t$$
$$w \geq 0$$

So, its dual can be written as

$$\left.\begin{aligned} &\text{Minimize } Z' = (-c^t)^t x \\ &\text{Subject to } (-A^t)^t x \geq (-b^t)^t \\ &x \geq 0 \end{aligned}\right\} \quad \cdots (3)$$

~~But since Minimize $Z' = cx$~~

Now the dual (3) can be written as

$$\text{Minimize } Z' = -cx$$
$$\text{Subject to } -Ax \geq -b$$
$$x \geq 0$$

~~But know since Minimum $Z' = -\text{Maximum of }(-Z')$ $= -\text{Max}$~~

but $\text{Min } Z' = -\text{Max}(-Z') = -\text{Max } cx = -\text{Max } Z$

and $-Ax \geq -b$ is equivalent to $Ax \leq b$

So, the problem ultimately reduces to

$$\text{Maximize } Z = cx$$
$$\text{Subject to } Ax \leq b$$
$$x \geq 0$$

which is the primal (1).

Hence the theorem follows.

**Theorem 6.2** (Weak duality Theorem) : If $x_0$ be any feasible solution to the primal problem

$$\text{Max } Z = cx$$
$$\text{Subject to } Ax \leq b$$
$$x \geq 0$$

and $w_0$ be any feasible solution to the dual problem

$$\text{Min } W = b^t w$$
$$\text{subject to } A^t w \geq c^t$$
$$w \geq 0$$

then $cx_0 \leq b^t w_0$

Proof: As $w_0$ is a feasible solution of the dual, we have, $A^t w_0 \geq c^t$ or, $(A^t w_0)^t \geq (c^t)^t$ or, $w_0^t A \geq c$ ---(1)

Postmultiplying (1) by $x_0$, we have

$$(w_0^t A) x_0 \geq c x_0 \quad [\because x_0 \geq 0]$$

$$\text{or, } \quad w_0^t (A x_0) \geq c x_0$$

$$\text{or, } \quad w_0^t b \geq w_0^t (A x_0) \geq c x_0 \quad [\text{as } A x_0 \leq b]$$

$$\text{or, } \quad w_0^t b \geq c x_0$$

$$\text{or, } \quad b^t w_0 \geq c x_0 \quad (\text{As } b^t w_0 \text{ is a scalar})$$

$$\text{or, } \quad c x_0 \leq b^t w_0$$

Hence the theorem is proved.

**Theorem 6.3** If $x_0$ is a feasible solution to the primal problem

$$\text{Max } Z = cx$$
$$\text{Subject to } Ax \leq b$$
$$x \geq 0$$

and $w_0$ is a feasible solution to the dual problem

$$\text{Min } W = b^t w$$
$$\text{subject to } A^t w \geq c^t$$
$$w \geq 0$$

and $cx_0 = b^t w_0$, then $x_0$ and $w_0$ are the optimal feasible solutions of the primal and dual respectively

Proof: Let $x$ and $w$ be any two feasible solutions of the primal and dual respectively. Then by Theorem 6.2

$cx \leq b^t w$

Then $cx \leq b^t w_0$ (As $w_0$ is a feasible solution of the dual)

or, $cx \leq b^t w_0 = cx_0$

or, $cx \leq cx_0$

So, $x_0$ is an optimal feasible solution of the primal and $\max Z = cx_0$

In the same way, $cx_0 \leq b^t w$ (as $x_0$ is a feasible solution of the primal)

So, $b^t w_0 = cx_0 \leq b^t w$

or, $b^t w_0 \leq b^t w$

So, $w_0$ is an optimal feasible solution of the dual and $\min W = b^t w_0$. Hence the theorem is proved.

Theorem 6.3 (Complementary slackness)

For any pair of optimal solutions to the primal problem

Max $z = cx$

Subject to $Ax \leq b$, $x \geq 0$ $\quad \Bigg\} \cdots (1)$

and its dual ~~Min W = A^t w ≥ c^t, w≥0~~

Min $W = b^t w$

Subject to $A^t w \geq c^t$, $w \geq 0$ $\quad \Bigg\} \cdots (2)$

$w_i x_{n+i} = 0$, $i = 1, 2, \cdots m$; $x_j w_{m+j} = 0$, $j = 1, 2, \cdots, n$, i.e.,

(i) The product of ith legitimate or original variable of the dual and ith slack variable of the primal vanishes, $i = 1, 2, \cdots, m$. Also

(ii) The product of jth legitimate or original variable of the primal and jth surplus variables of the dual vanishes $j = 1, 2, \cdots n$

(This is called complementary slackness)

Proof: (i) After adding a set of m slack variables $x_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}$ to the constrains of (1) we have,

$$Ax + I_m x_s = b \quad \cdots (3)$$ [$I_m$ is the identity matrix of order $m$]

Premultiplying (3) by $w^t \geq 0$, we get

$$w^t(Ax) + w^t x_s = w^t b \quad \cdots (4)$$

Now if $[x_0, x_s]$ is an optimal solution of the primal and $w_0$ an optimal solution to the dual we have from Theorem 6.1 and 6.2