$\text{Max } Z = w_0^t (A x_0) = \text{Min } W = w_0^t b$ from which we get

$$w^t x_s = 0 \quad \cdots (5)$$

Then $\sum_{i=1}^{m} w_i \cdot x_{n+i} = 0$; Since $w_i \geq 0$ and $x_{n+i} \geq 0$

then $w_i x_{n+i} = 0$, $i = 1, 2, \ldots, m \quad \cdots (6)$

(ii) Again after adding a set of n surplus variables w_s from the constraints of (2), we get

$$A^t w - I_n w_s = c^t \quad (\because I_n \text{ is the identity matrix of order } n) \quad \cdots (7)$$

Premultiplying (7) by x^t, $x^t \geq 0$, we get

$$x^t (A^t w) - I_n x^t w_s = x^t c^t \quad \cdots (8)$$

Since all terms of (8) are scalars, we have

$$w^t (A x) - x^t w_s = cx \quad \cdots (9)$$

$$[\because x^t (A^t w) = (Ax)^t w = w^t (Ax)]$$

As x_0 and $[w_0, w_s]$ are the optimal solution of the primal and dual, then again, we have

$$w_0^t (A x_0) = c x_0 \Rightarrow x_0^t w_s = 0$$

Since $x_j \geq 0$, $w_{m+j} \geq 0$ then

$$x_j \cdot w_{m+j} = 0, \quad j = 1, 2, \ldots, n$$

Deduction: From (6) if $x_{n+i} \neq 0$, then $w_i = 0$ and conversely if $w_i \neq 0$ then.

$x_{n+i} = 0$. From the deduction, we can state theorem in the manner: (i) If a slack variable x_{n+i} is added to the ith constraint of equation (1) is different from zero in any optimal solution of (1) then the ith dual variable w_i of (2) will be zero in every optimal solution in (2), i.e., the ith dual constraint will be an equality. Conversely, (ii) If w_i is different from zero in any optimal solution of (2), then $x_{n+i} = 0$ in every optimal solution in (1), i.e., the ith primal constraints will be an equality.

We now state the fundamental theorem of duality without proof

Theorem 6.5 (Fundamental theorem of duality)

If either the primal

Max. $Z = Cx$

Subject to $Ax \leq b$

$x \geq 0$

or the dual Min. $W = b^t w$

subject to $A^t w \geq c^t$

$w \geq 0$

has a finite optimal solution, then the other problem will also have a finite optimal solution. Furthermore, the optimal values of the objective functions in both the problem will be same i.e., Max Z = Min W.

Note : If x_B be the optimal feasible solution to the primal problem corresponding to the optimal basis B and if c_B be the associated cost vector, then

$$x_B = B^{-1}b$$

and $w_0 = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}$ where $w_0^T = c_B B^{-1}$

is an optimal solution to the dual problem.

We state another two theorems without proof.

Theorem 6.6 : If either of the prime or dual has unbounded solution then the other will have no feasible solution.

Theorem 6.7 If the primal problem has feasible solution and the dual has no feasible solution then the primal problem has unbounded solution.

We mention all the results in a tabular form

Primal	Dual	Conclusion
unbounded solution of the primal (dual)	—	no feasible solution of the dual (primal)
no feasible solution of the primal (dual)	—	Either unbounded solution or no feasible solution of the dual (primal)
bounded optimal solution	—	Dual has also bounded optimal solution and vice versa
Feasible solution	Feasible solution	~~Feasible solution~~ Finite optimal values for both exist.
Feasible solution	No feasible solution	unbounded solution of the primal
no feasible solution	feasible solution	unbounded solution of the dual
no feasible solution	no feasible solution	no optimal solution to either problem

Note: The values of $z_j - c_j$ for the column corresponding to the slack (surplus) vectors in the final simplex table of the primal problem, are the values of the corresponding dual optimal variable and vice versa provided the problem is solved as a maximization problem. The result is very important to find out the optimal dual variables and vice versa.

Proof: $z_j - c_j = z_j \geq 0$ [as the prices of cost corresponding slack (surplus) variables ~~[illegible]~~ are zero.

So, $z_j - c_j = z_j = c_B y_j = c_B B^{-1} a_j$

$= c_B B^{-1} e_j$ (e_j is the jth column of I_m, $j = 1, 2, \dots, m$)

So, $z_j - c_j$ for the columns of the slack vectors are

$= c_B B^{-1} (e_1, e_2, \dots e_m)$

$= c_B B^{-1} I_m = c_B B^{-1} = w_0^t$ (from the note of page-103).

Importance of the Duality Theory

When the number of constraints are greater than the number of variables, duality theory is very helpful in solving the problem by simplex method. For example, let the number of constraints be five and number of variables be two in the primal problem. If we try to ~~sl~~ solve the primal problem by simplex method, then the basis will be a 5×5 square matrix it requires much