time to compute each table. But if we convert the into its dual we get only two constraints instead of 5 and the dual can be solved easily. Now solving the dual problem we get the optimal value of the objective function of primal as well as the primal variables optimal variables. Hence using duality theory we can solve the problems (sometimes) more easily and quickly. Duality theory has also much economical importances.

Worked Example 1. Write down the dual of the following problem and solving the dual problem by simplex method, find the optimal solutions and optimal values of the primal and dual as well:

$$\text{Maximize } z = 3x_1 + 4x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\le 10 \\ 2x_1 + 3x_2 &\le 18 \\ x_1 &\le 8 \\ x_2 &\le 6, \quad x_1, x_2 \geq 0 \end{aligned}$$

Solution: The dual problem is

$$\text{Minimize } W = 10w_1 + 18w_2 + 8w_3 + 6w_4$$

subject to

$$\begin{aligned} w_1 + 2w_2 + w_3 &\geq 3, \\ w_2 + 3w_2 + w_4 &\geq 4, \\ w_1, w_2, w_3, w_4 &\geq 0 \end{aligned}$$

To solve the problem by simplex method we have noted that unit matrix I_2 is already present there. So, ~~introducing~~ introducing surplus variables w_5 and w_6, the problem reduces to

Maximize $(-W) = -10w_1 - 18w_2 - 8w_3 - 6w_4 + 0\cdot w_5 + 0\cdot w_6$

Subject to
$$w_1 + 2w_2 + w_3 - w_5 = 3$$
$$w_1 + 3w_2 + w_4 - w_6 = 4$$
$$w_1, w_2, w_3, w_4, w_5, w_6 \geq 0$$

where Min $W = -$Max$(-W)$ with the same solution set

		c	−10	−18	−8	−6	0	0	
B	C_B	b	a_1	a_2	a_3	a_4	a_5	a_6	Min. ratio
a_3	−8	3	1	2	1	0	−1	0	3/2
a_4	−6	4	1	$\boxed{3}$	0	1	0	−1	4/3*
$z_j - c_j$		−48	−4	−16*	0	0	8	6	
a_3	−8	1/3	1/3	0	1	−2/3	−1	2/3	
a_2	−18	4/3	1/3	1	0	1/3	0	−1/3	
$z_j - c_j$		−80/3	4/3	0	0	16/3	8	2/3	

In the 2nd table all $z_j - c_j \geq 0$, $j = 1,2,3,4,5,6$.

Thus we reach at the optimal stage.

So, Min $W = -$Max$(-W) = -\left(-\frac{80}{3}\right) = \frac{80}{3}$

at $w_1 = 0, w_2 = 4/3, w_3 = 1/3$ and $w_4 = 0$ (w_1, w_4 are non basic variables). Thus using the property of duality, Max Z = Min W = $80/3$ at $x_1 = 8$ and $x_2 = 2/3$ which are the $z_j - c_j$ corresponding surplus variables w_5 and w_6 respectively.

We now prove two theorems:

<u>Theorem 6.8</u> If the kth constraint of a primal be an equation, the kth dual variable will be unrestricted in sign.

Proof: As the kth constraint of the primal is an equation, therefore the primal in the standard form can be written as

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$
$$\cdots\cdots\cdots$$
$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n \le b_k$$
$$-a_{k1}x_1 - a_{k2}x_2 - \cdots - a_{kn}x_n \le -b_k$$
$$a_{k+1\,1}x_1 + a_{k+1\,2}x_2 + \cdots + a_{k+1\,n}x_n \le b_{k+1}$$
$$\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m,$$
$$x_1, x_2, \ldots, x_n \ge 0$$

(As $x = k$ can be written as $x \le k$ and $-x \le -k$)

The dual of the above primal can be written as

$$\text{Minimize } W = b_1w_1 + b_2w_2 + b_k(w_k' - w_k'') + b_{k+1}w_{k+1} + \cdots + b_mw_m$$

subject to

$$a_{11}w_1 + a_{21}w_2 + \cdots + a_{k1}(w_k' - w_k'') + a_{k+1\,1}w_{k+1} + \cdots + a_{m1}w_m \ge c_1$$
$$a_{12}w_1 + a_{22}w_2 + \cdots + a_{k2}(w_k' - w_k'') + a_{k+1\,2}w_{k+1} + \cdots + a_{m2}w_m \ge c_2$$
$$\cdots\cdots\cdots$$
$$a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{kn}(w_k' - w_k'') + a_{k+1\,n}w_{k+1} + \cdots + a_{mn}w_m \ge c_n$$

Putting $w_k = w_k' - w_k''$, the dual can be written as

$$\text{Minimize } W = b_1w_1 + b_2w_2 + \cdots + b_kw_k + \cdots + b_mw_m$$

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Subject to

$$a_{11}w_1 + a_{21}w_2 + \cdots + a_{k1}w_k + \cdots + a_{m1}w_m \geq c_1$$
$$a_{12}w_1 + a_{22}w_2 + \cdots + a_{k2}w_k + \cdots + a_{m2}w_m \geq c_2$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{kn}w_k + \cdots + a_{mn}w_m \geq c_n$$

and $w_1, w_2, \ldots, w_{k-1}, w_{k+1}, \ldots, w_m \geq 0$ but w_k is unrestricted in sign, i.e., it can take any real values as the difference of two non-negative variables w_k' and w_k''. Hence the theorem is proved.

<u>Theorem 6.9</u> If any variable of the primal problem be unrestricted in sign, the corresponding dual constraints will be strictly an equation.

Proof: Let the kth primal variable x_k be unrestricted in sign. Then putting $x_k = x_k' - x_k''$ where x_k' and x_k'' are non-negative variables, the primal problem can be written as

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \cdots + c_k(x_k' - x_k'') + \cdots + c_nx_n$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}(x_k' - x_k'') + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}(x_k' - x_k'') + \cdots + a_{2n}x_n \leq b_2$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mk}(x_k' - x_k'') + \cdots + a_{mn}x_n \leq b_m$$

and $x_1, x_2, \ldots x_k', x_k'', x_{k+1}, \ldots, x_n \geq 0$

The dual of the above problem can be written