as

$$\text{Minimize } W = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m$$

Subject to

$$a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m \geq c_1$$

$$a_{12} w_1 + a_{22} w_2 + \cdots + a_{m2} w_m \geq c_2$$

$$\cdots \quad \cdots \quad \cdots$$

$$\begin{cases} a_{1k} w_1 + a_{2k} w_2 + \cdots + a_{mk} w_m \geq c_k \\ -a_{1k} w_1 - a_{2k} w_2 - \cdots - a_{mk} w_m \geq -c_k \end{cases}$$

$$\cdots \quad \cdots \quad \cdots$$

$$a_{1n} w_1 + a_{2n} w_2 + \cdots + a_{mn} w_m \geq c_n$$

and $w_1, w_2, \ldots, w_m \geq 0$

The two ~~contrs~~ constraints under the bracket $\{\ \}$ are equivalent to an equation

$$a_{1k} w_1 + a_{2k} w_2 + \cdots + a_{mk} w_m = c_k$$

Hence the kth dual constraint is an equation which proves the theorem.

<u>Example 1</u> Find the dual of the following primal problem: maximize $Z = x_1 + 4x_2 + 3x_3$

Subject to $2x_1 + 3x_2 - 5x_3 \leq 2$

$$3x_1 - x_2 + 6x_3 \geq 1$$

$$x_1 + x_2 + x_3 = 4$$

$x_1, x_2 \geq 0$, x_3 is unrestricted in sign.

Solution: Since x_3 is unrestricted in sign then put $x_3 = x_3' - x_3''$, $x_3', x_3'' \geq 0$ and the third constraint is an equation which can be written in the manner

$$x_1+x_2+x_3 \le 4$$

and $$x_1+x_2+x_3 \ge 4$$

So, the primal problem after putting the value of x_3 is

$$\text{maximize } Z = x_1+4x_2+3x_3'-3x_3''$$

subject to
$$2x_1+3x_2-5x_3'+5x_3'' \le 2$$
$$-3x_1+x_2-6x_3'+6x_3'' \le -1$$
$$x_1+x_2+x_3'-x_3'' \le 4$$
$$-x_1-x_2-x_3'+x_3'' \le -4,$$
$$x_1, x_2, x_3', x_3'' \ge 0$$

So, the dual problem is

$$\text{Minimize } W = 2w_1-w_2+4(w_3-w_4)$$

subject to
$$2w_1-3w_2+(w_3-w_4) \ge 1$$
$$3w_1+w_2+(w_3-w_4) \ge 4$$
$$\left.\begin{array}{l} -5w_1-6w_2+(w_3-w_4) \ge 3 \\ 5w_1+6w_2-(w_3-w_4) \ge -3, \end{array}\right\}$$
$$w_1, w_2, w_3, w_4 \ge 0$$

Putting $w_3-w_4 = w_3'$ (which is unrestricted in sign) the dual problem is

$$\text{Minimize } W = 2w_1-w_2+4w_3'$$

subject to
$$2w_1-3w_2+w_3' \ge 1$$
$$3w_1+w_2+w_3' \ge 4$$
$$-5w_1-6w_2+w_3' = 3,$$
$w_1, w_2 \ge 0$ and w_3' is unrestricted in sign.

7. Transportation and Assignment Problem

7.1: Transportation Problem and its introduction: Transportation problem is a problem of linear programming. The object of this problem is to transport various amounts of a single homogeneous commodity, initially stored at different sources (or origins), to different destinations in such a way that the cost of transportation is minimum. For this problem, the following information are to be needed:

(1) Available amount of the commodity at different origins.
(2) Amounts demanded at different destinations.
(3) The transportation cost of one unit of commodity from various origin to various destinations.

7.2. Mathematical formulation of transportation problem: Let, in a transportation problem, there be m origins $O_1, O_2, \ldots, O_m$ with available quantities of commodity $a_1, a_2, \ldots, a_m$ and n destinations $D_1, D_2, \ldots, D_n$ with demands $b_1, b_2, \ldots, b_n$. Let c_{ij}, $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$, be the transportation cost to transfer one unit of the commodity from the ith origin to the jth destination. Our problem is to determine x_{ij}, the amount of units to be transferred from the ith origin to the jth destination, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, so that the total transportation cost $\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$ be minimum. If it is assumed that the total availability is equal to the total demand, i.e., $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, then the transportation problem is called a balanced transportation problem.

The problem can be put in a tabular form as follows:

Destinations

ORIGINS	D_1	D_2	D_3	$\ldots$	D_n	
O_1	$\boxed{x_{11}}$ c_{11}	$\boxed{x_{12}}$ c_{12}	$\boxed{x_{13}}$ c_{13}	$\ldots$	$\boxed{x_{1n}}$ c_{1n}	a_1
O_2	$\boxed{x_{21}}$ c_{21}	$\boxed{x_{22}}$ c_{22}	$\boxed{x_{23}}$ c_{23}	$\ldots$	$\boxed{x_{2n}}$ c_{2n}	a_2
O_3	$\boxed{x_{31}}$ c_{31}	$\boxed{x_{32}}$ c_{32}	$\boxed{x_{33}}$ c_{33}	$\ldots$	$\boxed{x_{3n}}$ c_{3n}	a_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
O_m	$\boxed{x_{m1}}$ c_{m1}	$\boxed{x_{m2}}$ c_{m2}	$\boxed{x_{m3}}$ c_{m3}	$\ldots$	$\boxed{x_{mn}}$ c_{mn}	a_m
	b_1	b_2	b_3	$\ldots$	b_n	

Thus a transportation problem can be mathematically put as a linear programming problem as follows:

$$\text{Minimize } Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to
$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots, m$$
$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots, n$$
$$x_{ij} \geq 0 \quad \text{for all } i \text{ and } j$$

If the transportation is a balanced transportation problem then $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.