Theorem 1 The number of basic variables in a balanced transportation problem is at most $(m+n-1)$

Proof: In a transportation problem with m origins and n destinations, total number of constraints are m+n and they are given by

$$\sum_{j=1}^{n} x_{ij} = a_i \ , \ i=1,2,\dots,m \quad \dots (1)$$

and

$$\sum_{i=1}^{m} x_{ij} = b_j \ , \ j=1,2,\dots,n \quad \dots (2)$$

Again, since this is a balanced transportation problem,

so

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad \dots (3)$$

Now from (1) using (3), we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \quad \dots (4)$$

With the help of (4), we will show that one constraint is redundant, i.e., one constraint can be obtained from the other constraints. Summing the first $(n-1)$ constraints of (2), we get

$$\sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j \quad \dots (5)$$

Subtracting (5) from (4), we get

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} x_{ij}\right) = \sum_{j=1}^{n} b_j - \sum_{j=1}^{n-1} b_j$$

or, $\sum_{i=1}^{m} x_{in} = b_n$ which is nth constraint.

of (2). Thus the number of basic variables of a transportation problem is at most $m+n-1$.

Note 1 We can prove that the number of basic variables of transportation problem is exactly $m+n-1$

Note 2 Here $a_i \geq 0$ and $b_j \geq 0$, $i=1,2,\ldots,m$ $j=1,2,\ldots,n$

Note 3: Henceforth, we will write transportation problem as TP

Theorem 2 (Existence of a feasible solution)

A necessary and sufficient condition for the existence of a feasible solution to a TP is $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Proof: The condition is necessary:

Let the TP has a feasible solution. Then we get

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i \quad \text{and}$$

$$\sum_{j=1}^{n}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n} b_j \quad \text{and then } \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

The condition is sufficient:

Let $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = M$ (say)

We shall prove that $x_{ij} = \frac{a_i b_j}{M}$, for all i and j, will be a feasible solution of the TP.

Now $$\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \frac{a_i b_j}{M} = \frac{b_j \sum_{i=1}^{m} a_i}{M} = b_j \frac{M}{M} = b_j, \quad j=1,2,\ldots,n$$

and $$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \frac{a_i b_j}{M} = \frac{a_i}{M}\sum_{j=1}^{n} b_j = a_i \cdot \frac{M}{M} = a_i, \quad i=1,2,\ldots,m$$

So, $x_{ij} = \frac{a_i b_j}{M}$, $i=1,2,\dots,m$, $j=1,2,\dots,n$ satisfy all the constraints of the TP. Also $x_{ij} \geq 0$ for all i and j as $a_i > 0$ and $b_j > 0$ for all i and j.

Thus $x_{ij} = \frac{a_i b_j}{M}$, $i=1,\dots,m$ $j=1,\dots,n$ is a feasible solution of the TP.

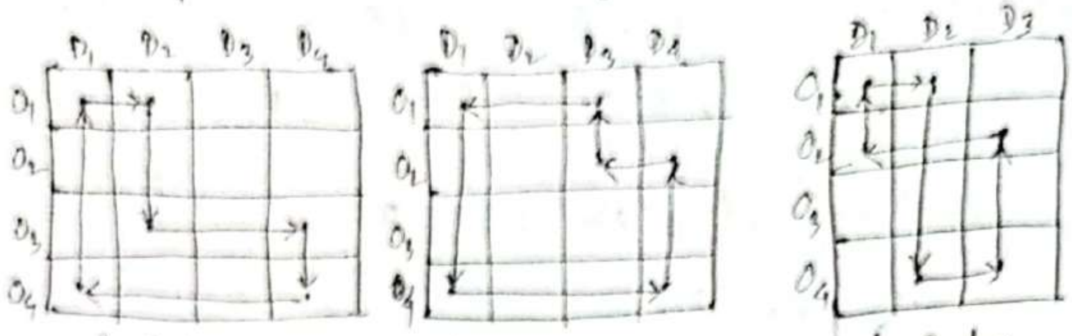
Theorem 3 The solution of a balanced TP is never unbounded.

Proof: From Theorem 2, there exists a feasible solution to the TP. As all c_{ij} and x_{ij} are finite quantities, then Z is a bounded function. Hence the problem has an optimal value and there exists a feasible solution which will be an optimal solution to the TP.

7.3. Loops in a transportation problem

In a transportation table, an ordered set of four or more cells are said to form a loop if and only if two consecutive cells in the ordered set lie either in the same row or in the same column and if the first and the last cell of the set also lie either in the same row or in the same column.

Some loops are shown in the following figures :



Loop L_1 Loop L_2 Loop L_3

Loop $L_1 = \{(1,1),(1,2),(2,3),(3,4),(4,4),(4,1)\}$

Loop $L_2 = \{(1,1),(4,1),(4,4),(2,4),(2,3),(1,3)\}$

Loop $L_3 = \{(1,1),(1,2),(4,2),(4,3),(2,3),(2,1)\}$

<u>NOTE 1</u> If a set of cells in a TP contains a loop then we can prove that corresponding columns are linearly dependent. So, if we chose cells which do not contain loop, then their corresponding columns are linearly independent.

<u>NOTE 2</u> To find a basic feasible solution, we choose $(m+n-1)$ cells in such a way that those cells do not contain any loop. Then the variables related to the corresponding cells should be basic variables. So, if we can find out $x_{ij} \geq 0$ for $(m+n-1)$ cells which do not contain any loop, then we get a basic feasible solution, where those $(m+n-1)$ components are basic components