For matrix B_2, x_1 and x_3 are basic components and non-basic component is x_2 and $x_2 = 0$

So, $\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = B_2^{-1} b = \frac{1}{34}\begin{bmatrix} 7 & 2 \\ -10 & 2 \end{bmatrix}\begin{bmatrix} 10 \\ 33 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

So, $x_1 = 4$, $x_2 = 0$ and $x_3 = -1$ is a basic solution. As $x_3 = -1 < 0$, this is not a basic feasible solution.

For matrix B_3, x_2 and x_3 are basic variables and x_1 is the non-basic variable and $x_1 = 0$.

So, $\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = B_3^{-1} b = \frac{1}{34}\begin{bmatrix} 7 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 10 \\ 33 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

So, $x_1 = 0$, $x_2 = 4$, $x_3 = 3$ is a basic solution. As $x_i \geq 0$, for $i = 1,2,3$, so it is a basic feasible solution

Note when the Basis matrix is $B_1 = [a_1, a_2] = \begin{bmatrix} 2 & 4 \\ 10 & 3 \end{bmatrix}$, then the basic variables are x_1 and x_2. So, non-basic variable is x_3 and $x_3 = 0$. ~~So~~ So, putting $x_3 = 0$ in the system, we have $2x_1 + 4x_2 = 10$ or, $2x_1 + 4x_2 - 10 = 0$

and $10x_1 + 3x_2 = 33$ or, $10x_1 + 3x_2 - 33 = 0$

Solving, $\frac{x_1}{-132+30} = \frac{x_2}{-100+66} = \frac{1}{6-40}$ or, $\frac{x_1}{-102} = \frac{x_2}{-34} = \frac{1}{-34}$

or, $x_1 = 3$, $x_2 = 1$. In this way, we can get the basic solution $x_1 = 3$, $x_2 = 1$, $x_3 = 0$ without finding the B_1^{-1}. Only we have to check that B_1 is a basis matrix. We can get the other basic solutions in the same way.

Example 2

A system of linear equation is given by

$$4x_1 + 2x_2 + 3x_3 - 8x_4 = 6$$
$$3x_1 + 5x_2 + 4x_3 - 6x_4 = 8$$

(a) How many basic solutions are there ?

(b) Find all of them

(c) Discuss the nature of each and every basic solution.

Solution: (a) Here the system of linear equation can be written as $Ax = b$, where $A = [a_1, a_2, a_3, a_4] = \begin{bmatrix} 4 & 2 & 3 & -8 \\ 3 & 5 & 4 & -6 \end{bmatrix}$

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, and $b = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$, $a_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $a_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ $a_3 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $a_4 = \begin{pmatrix} -8 \\ -6 \end{pmatrix}$

As, $\begin{vmatrix} 4 & 2 \\ 3 & 5 \end{vmatrix} = 20 - 6 = 14 \neq 0$, so rank of $A = r(A) = 2$.

So, maximum possible number of basic solution is ${}^4C_2 = \frac{4 \times 3}{2} = 6$ as there are 6 square submatrices of order 2.

Let $B_1 = (a_1, a_2) = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix}$, $B_2 = (a_1, a_3) = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$

$B_3 = (a_1, a_4) = \begin{bmatrix} 4 & -8 \\ 3 & -6 \end{bmatrix}$, $B_4 = (a_2, a_3) = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$

$B_5 = (a_2, a_4) = \begin{bmatrix} 2 & -8 \\ 5 & -6 \end{bmatrix}$ and $B_6 = (a_3, a_4) = \begin{bmatrix} 3 & -8 \\ 4 & -6 \end{bmatrix}$

Now $\det B_1 = 14 \neq 0$, $\det B_2 = 7 \neq 0$, $\det B_3 = 0$, $\det B_4 = -7 \neq 0$, $\det B_5 = 28 \neq 0$ and $\det B_6 = 14 \neq 0$. Here B_3 is not a Basis matrix but B_1, B_2, B_4, B_5, and B_6 are five basis matrices, So, the system has 5 basic solutions

(b) Taking B_1 as the matrix, basic variables are x_1, x_2 and non-basic variable are x_3 and x_4 and $x_3 = 0$, and $x_4 = 0$. So we have

$$4x_1 + 2x_2 = 6$$
$$3x_1 + 5x_2 = 8$$

So, we have $x_1 = 1$, $x_2 = 1$. So, $y_1 = (1, 1, 0, 0)$ is a basic solution

For the basis matrix, B_2, non-basic variables are x_2 and x_4 and $x_2 = x_4 = 0$ and the basic variables x_1 and x_3 are given by

$$4x_1 + 3x_3 = 6$$
$$3x_1 + 4x_3 = 8$$

Solving, we get, $x_1 = 0$, $x_3 = 2$

So, $y_2 = (0, 0, 2, 0)$ is a basic solution

For basis matrix B_4, nonbasic variables are x_1, x_4 and $x_1 = x_4 = 0$ and the basic variables x_2 and x_3 are given by

$$2x_2 + 3x_3 = 6$$
$$5x_2 + 4x_3 = 8$$

Solving, we get $x_2 = 0$, $x_3 = 2$

So, $y_4 = (0, 0, 2, 0)$ is a basic solution

For basis matrix B_5, non-basic variables are x_1 and x_3 and $x_1 = x_3 = 0$ and the basic variable x_2 and x_4 are given by

$$2x_2 - 8x_4 = 6$$
$$5x_2 - 6x_4 = 8$$

Solving, we get $x_2 = 1$, $x_4 = -\frac{1}{2}$

So, $y_5 = \left(0, 1, 0, -\frac{1}{2}\right)$ is a basic solution

For the basis matrix B_6, non-basic variables are x_1, x_2 and $x_1 = x_2 = 0$ and the basic variables x_3, x_4 are given by

$$3x_3 - 8x_4 = 6$$
$$4x_3 - 6x_4 = 8$$

Solving, we get $x_3 = 2$, $x_4 = 0$

So, $y_6 = (0, 0, 2, 0)$ is a basic solution.

(c) Here y_1, y_2, y_4 and y_6 are basic feasible solutions as all the components of the solutions are non-negative.

But y_5 is not a basic feasible solution

as $x_4 = -\frac{1}{2} < 0$

Here y_2 is a degenerate basic feasible solution as the basic component $x_1 = 0$.

y_4 is a degenerate basic feasible solution as the basic variable $x_2 = 0$

y_6 is a degenerate basic feasible solution as the basic variable $x_4 = 0$

Here y_1 is a non-degenerate basic feasible solution as no basic component is zero.

Similarly y_5 is a non-degenerate basic solution though it is not feasible.

<u>Note</u> Henceforth, we will write for basic solution BS and for basic feasible solution BFS.

<u>Exercises</u> 1. Show that $x_1 = 5, x_2 = 0, x_3 = -1$ is a basic solution of the system of equations

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5$$

Find the other basic solutions, if any there be any.

2. Determine all the basic feasible solutions of the set of equations

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$
$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$