2.7 Matrix formulation of LPP

From the discussion of mathematical formulation of Linear Programming Problems (LPP), a general linear programming problem can be stated mathematically as follows:

Find out a set of values $x_1, x_2, \ldots, x_n$ which will optimize (either maximize or minimize) the linear function

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to the restrictions

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ (\leq = \geq)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ (\leq = \geq)\ b_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ (\leq = \geq)\ b_m \end{array}\right\} \quad (1)$$

and the non-negative restrictions $x_j \geq 0$, $j = 1, 2, \ldots, n$

where a_{ij}, c_j and b_i's ($i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$) are all constants and x_j's are variables, $j = 1, 2, \ldots, n$. Each of the linear expressions on the left side connected to the corresponding constants on the right side, by only one of the signs '$\leq$', '$=$' and '$\geq$', is known as a constraint. A constraint is either an equation or an inequation. $x_j \geq 0$, $j = 1, 2, \ldots, n$ are called non-negativity restrictions.

The linear function $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is known as objective function.

By using the matrix notation we can write down the problem as follows:

Optimize $z = cx$ subject to the restrictions

$$Ax \ (\leq = \geq) b, \quad x \geq 0 \quad \ldots (2)$$

where $A = [a_{ij}]_{m \times n}$ ~~coefficient~~ is the coefficient matrix.

$c = (c_1, c_2, \ldots, c_n)$ is known as cost or price vector as it determines the cost of production or the price of a commodity.

$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is decision variable vector and

$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ is known requirement vector and $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is the null vector with n components

(Note for two n-component vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ $x \leq y$ means $x_j \leq y_j, j = 1, 2, \ldots n$)

If all the ~~components~~ constraints are equations then the LPP ~~can~~ can be written as

$$\text{Optimize } z = cx$$
$$\text{subject to } Ax = b, \ x \geq 0 \quad \ldots (3)$$

This form is called the standard form of an LPP.

<u>Feasible solution to an LPP</u>: A set of values of the variables which satisfy all the constraints and all the non-negativity restrictions of variables, is known as a feasible solution (FS) to the LPP.

<u>Optimal solution to an LPP</u>: A feasible solution to an LPP which optimizes the objective function is known as an optimal solution to the LPP.

3. Euclidean space, Convex set etc.

In our earlier linear algebra class, we have seen that the vector space $\mathbb{R}^n$ with usual inner product is an euclidean space. Any n-component row vector or column vector can be treated as a member of $\mathbb{R}^n$.

(Note 1: Sometimes E^n is written for the euclidean space $\mathbb{R}^n$)

(Note 2: A n component row vector is written as $(a_1, a_2, \ldots, a_n)$ and an n-component column vector is written as $[b_1, b_2, \ldots, b_n]$)

Hyperplane: A hyperplane in $\mathbb{R}^n$ is defined to be the set $X = \{x \in \mathbb{R}^n : cx = z\}$ where $c \neq 0$, being a given n-component row vector and $z \in \mathbb{R}$ and cx is the usual inner product in $\mathbb{R}^n$. Here $x \in \mathbb{R}^n$ is written as an n-component column vector.

Examples:

1. $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 3x_1 + 2x_2 + 5x_3 = 5\}$ is a hyperplane in $\mathbb{R}^3$

2. $X = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 2x_1 + 5x_2 + 7x_3 + x_4 = 6\}$ is a hyperplane in $\mathbb{R}^4$.

A hyperplane passes through $0 = (0, 0, \ldots, 0)$ if and only if $z = 0$. Then $cx = 0$

So, we see that c is orthogonal to every vector x on the hyperplane and hence we can say that c is normal to the hyperplane. If $z \neq 0$ and x_1 and x_2 are any two distinct points lying on the hyperplane, then

$cx_1 - cx_2 = c(x_1 - x_2) = z - z = 0$ and c is orthogonal to $x_1 - x_2$ (x_1, x_2 being on the hyperplane). In $\mathbb{R}^2$ and $\mathbb{R}^3$ $x_1 - x_2$ is parallel to the line and the plane respectively.

Thus, even with $z \neq 0$, we can say that c is normal to the hyperplane. The two vectors $\frac{\pm c}{|c|}$ are unit normals to the hyperplane.

The hyperplanes having the unit normals are said to be parallel. Moving a hyperplane $cx = z$ parallel to itself is accomplished by increasing or decreasing ~~it~~ the value of z.

The hyperplane $cx = z$ in $\mathbb{R}^n$ divides whole of $\mathbb{R}^n$ into three mutually disjoint sets as

$$X_1 = \{x \in \mathbb{R}^n : cx < z\}$$
$$X = X_2 = \{x \in \mathbb{R}^n : cx = z\}$$
$$\text{and } X_3 = \{x \in \mathbb{R}^n : cx > z\}.$$

X_1 and X_3 as defined above are called open halfspaces corresponding to the hyperplane X.

Also $X_4 = \{x \in \mathbb{R}^n : cx \leq z\}$ and $X_5 = \{x \in \mathbb{R}^n : cx \geq z\}$ are called closed half spaces corresponding to the hyperplane X.

Example: $x = [1, 2, 3, 4]$ lies on the open half space $cx > z$ given by the hyperplane $2x_1 + 3x_2 + 4x_3 + 5x_4 = 7$