Since $2\times1+3\times2+4\times3+5\times4 = 40 > 7$

But $x = [1,2,3,-4]$ lie in the open half space $cx < z$

given by the hyperplane $2x_1+3x_2+4x_3+5x_4 = 7$

(Here $c = (2,3,4,5)$, $x = [x_1, x_2, x_3, x_4]$ and $z = 7$)

It should be noted that in an LPP

optimize $z = cx$

subject to $Ax \ (\leq = \geq) b$, $x \geq 0$;

the objective function and also the constraints with equality sign represent hyperplanes. The constraints with signs $\leq$ or $\geq$ are closed half spaces produced by the hyperplanes with the sign of equality only.

A line passing through x_1, x_2 $(x_1 \neq x_2)$ in $\mathbb{R}^n$ is the set $X = \{x \in \mathbb{R}^n : x = \lambda x_2 + (1-\lambda)x_1, \ \lambda \in \mathbb{R}\}$

The set $X = \{x \in \mathbb{R}^n : x = \lambda x_2 + (1-\lambda)x_1, \ 0 \leq \lambda \leq 1\}$ is defined to be the line segment joining x_1 and x_2 in $\mathbb{R}^n$.

<u>convex set</u>: A set $X \subseteq \mathbb{R}^n$ is said to be a convex set if for any two points $x_1, x_2 \in X$, $x = \lambda x_1 + (1-\lambda)x_2 \in X$ for $0 \leq \lambda \leq 1$. (we can take also $x = \lambda x_1 + (1-\lambda)x_2$, $0 \leq \lambda \leq 1$)

So, a set $X \subseteq \mathbb{R}^n$ is convex if for any two points x_1 and x_2 in X, the line segment joining these two points is also in the set X.

Example 1: The set $X = \{(x,y) : x^2+y^2 \leq 4\}$ is a convex set in $\mathbb{R}^2$

Example 2: Geometrically, the shaded regions bounded by polygons shown in figure A are convex sets in $\mathbb{R}^2$

Figure A

Example 3 Geometrically, the shaded region shown in Figure B are not convex sets.

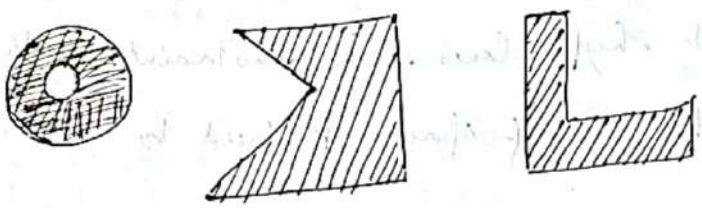

Figure B

3.1 Some results

(a) A hyperplane is a convex set in $\mathbb{R}^n$

Proof Let $X = \{x \in \mathbb{R}^n : cx = z\}$ be a hyperplane

Let x_1, x_2 be two points in X. Then

$$cx_1 = z \quad \text{and} \quad cx_2 = z \quad \cdots \quad (1)$$

Now let $x = \lambda x_2 + (1-\lambda)x_1, \quad 0 \le \lambda \le 1$

Then $cx = c(\lambda x_2 + (1-\lambda)x_1)$

$= \lambda(cx_2) + (1-\lambda)(cx_1)$

$= \lambda z + (1-\lambda)z$, by (1)

$= z$. So, $x \in X$

So, by definition, X is a convex set.

So, a hyperplane is a convex set.

(b) Let $\{X_\alpha : \alpha \in I\}$ be a family of convex sets in $\mathbb{R}^n$.

Then $X = \bigcap_{\alpha \in I} X_\alpha$ is also a convex set

Proof: Let $x_1, x_2 \in X$. Then $x_1, x_2 \in X_\alpha$ for each $\alpha \in I$.

So, $x = \lambda x_2 + (1-\lambda)x_1 \in X_\alpha$ for ~~each~~ each $\alpha \in I$, $0 \le \lambda \le 1$

as each X_α is convex set for $\alpha \in I$.

So, $x = \lambda x_2 + (1-\lambda)x_1 \in \bigcap_{\alpha \in I} X_\alpha$, for $0 \le \lambda \le 1$

Hence $\bigcap_{\alpha \in I} X_\alpha$ is a convex set

Note: In an LPP, the set of feasible solutions is given by $Ax (\le = \ge)$, $x \ge 0$ which is the ~~finite~~ ~~intersection~~ intersection of a finite number of hyperplanes or half spaces or both as given by the constraints. So, by the above result (b), it is a convex set.

Definition: Convex combination: A convex combination of a finite number of points $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$ is a point

$$x = \sum_{i=1}^{m} \lambda_i x_i, \quad \lambda_i \ge 0, \ i=1,2,\ldots,m \text{ and } \sum_{i=1}^{m} \lambda_i = 1$$

(c) The set of all convex combinations of a finite number of points $x_1, x_2, \ldots, x_m \in \mathbb{R}^n$ is a convex set, that is, the set $X = \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{m} \lambda_i x_i, \ \lambda_i \ge 0, \ i=1,2,\ldots,m \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \right\}$ is convex.

Proof: Let $v, w \in X$

Then $v = \sum_{i=1}^{m} \lambda_i' x_i$, $\lambda_i' \geq 0$, $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} \lambda_i' = 1$ —(A)

and $w = \sum_{i=1}^{m} \lambda_i'' x_i$, $\lambda_i'' \geq 0$, $i = 1, 2, \ldots, m$ and $\sum_{i=1}^{m} \lambda_i'' = 1$ —(B)

Now for λ such that $0 \leq \lambda \leq 1$

$$\lambda v + (1-\lambda) w$$

$$= \lambda\left(\sum_{i=1}^{m} \lambda_i' x_i\right) + (1-\lambda)\left(\sum_{i=1}^{m} \lambda_i'' x_i\right)$$

$$= \sum_{i=1}^{m} \left(\lambda \lambda_i' + (1-\lambda)\lambda_i''\right) x_i$$

$$= \sum_{i=1}^{m} \mu_i x_i \quad \text{where } \mu_i = \lambda \lambda_i' + (1-\lambda)\lambda_i'',\ i = 1, 2, \ldots, m.$$

Now as $0 \leq \lambda \leq 1$, $\lambda_i' \geq 0$ and $\lambda_i'' \geq 0$, $i = 1, 2, \ldots m$,

So $\mu_i \geq 0$, $i = 1, 2, \ldots, m$

Also $\sum_{i=1}^{m} \mu_i = \sum_{i=1}^{m} \left(\lambda \lambda_i' + (1-\lambda)\lambda_i''\right)$

$$= \lambda \sum_{i=1}^{m} \lambda_i' + (1-\lambda) \sum_{i=1}^{m} \lambda_i''$$

$$= \lambda + (1-\lambda) \quad \text{from (A) and (B)}$$

$$= 1$$

Hence $\lambda v + (1-\lambda) w \in X$ for $0 \leq \lambda \leq 1$.

So, X is a convex set.

Example: The set of all convex combination of three points x_1, x_2, x_3 in $\mathbb{R}^2$ is the following shaded region