which is the polygon connecting x_1, x_2 and x_3 and its interior

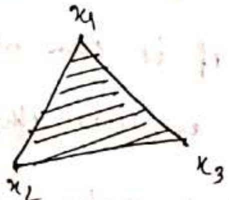


Figure 2

3.2 Cone: A cone $C \subseteq \mathbb{R}^n$ is a set of points with the following property: If x in C then μx is also in C, for all $\mu \geq 0$

The cone generated by a non-empty set X is the set

$$C = \{y \in \mathbb{R}^n : y = \mu x,\ \mu \geq 0 \text{ and } x \in X\}$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, a cone is often identical with the usual geometrical concept of a cone.

Vertex: The point $O = (0, 0, \dots, 0)$ is an element of any cone and is called the vertex of the cone.

Example: The figure D shows a cone in $\mathbb{R}^3$ generated by the set of points $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1,\ x_3 = 1\}$

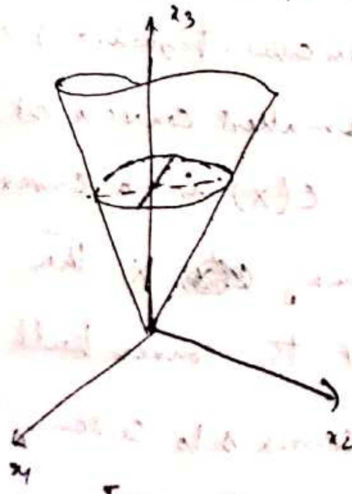
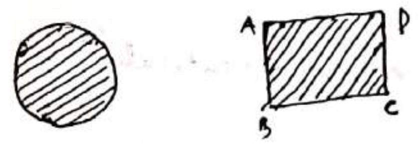


Figure - D

(b) Extreme point of a convex set : A point x of a convex set X is said to an extreme point if it can not be expressed as the convex combination of any two other distincts points in X. i.e., x can not be expressed as
$x = \lambda x_1 + (1-\lambda)x_2$, $0<\lambda<1$ for two points $x_1, x_2 \in X$.

Examples: 1. Every point of the boundary of circle in $\mathbb{R}^2$ is extreme point of the convex set which includes the boundary and the interior of the circle.
2. The four vertices of a rectangular region are extreme points of that region



Convex hull: Let X be a non-empty set in $\mathbb{R}^n$.
Then $C(X) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : x_i \in X, i=1,2,\dots,k, \ k \text{ is a finite natural number and } \lambda_i \geq 0, i=1,2,\dots,k \text{ and } \sum_{i=1}^{k}\lambda_i = 1 \right\}$

is called the convex hull of the given set X. i.e it is the set of finite convex combinations of points from X.
For $X = \{x_1, x_2, \dots, x_m\}$, $C(X)$ is called a convex polyhedron (we have proved earlier in page-31, at 3.1 (c) that it is a convex set) (i.e., if X be finite, then $C(X)$ is called a polyhedron)
It can be proved that $C(X)$ is the smallest convex set containing X in the sense that $C(X)$ is a convex set and if P be a convex set containing X then $C(X) \subset P$. Also we can show that the convex hull $C(X)$ of X is the intersection of all convex sets containing X

Examples: 1. The convex hull of the set $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ is $C(A) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ i.e., convex hull of the circle $x_1^2 + x_2^2 = 1$ is the circle $x_1^2 + x_2^2 = 1$ and its interior.

2. The convex hull of two points $x_1, x_2 \in \mathbb{R}^n$ is the set $X = \{x \in \mathbb{R}^n : \lambda x_1 + (1-\lambda) x_2,\ 0 \leq \lambda \leq 1\}$, i.e., the line segment joining x_1 and x_2 in $\mathbb{R}^n$

~~Supporting Hyperplane~~ (d): Simplex : A simplex is an n-dimensional convex polyhedron having exactly $(n+1)$ vertices.

Example: A simplex in zero dimension is a point; in one dimension, it is a line segment; ~~and~~ in two dimension, it is a triangle with its interior and in the three dimension, it is a tetrahedron with its interior.

(e) Supporting Hyperplane : Given a boundary point w of a convex set X, the hyperplane $cx = z$ is called a supporting hyperplane if $cw = z$ and if all of X lies in one closed half-spaces produced by the hyperplane, that is, $cu \geq z$ for all $u \in X$ or $cu \leq z$ for all $u \in X$.

There is a theorem which states that if w is a boundary point of a closed convex set, then there is at least one supporting hyperplane at w.

~~Considering~~ Considering the standard LPP of page-26, ~~if~~ also if the case of maximization taken, then the standard LPP is

$$\text{Maximize } z = cx$$

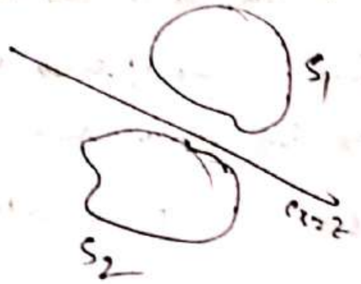
$$\text{Subject to } Ax = b$$

If X be the convex set of all feasible solution, then if x_0 be an optimal solution, then $cx_0 = z$ and $cu \leq cx_0 = z$ for all $u \in X$. So, $cx = z$ is then a supporting hyperplane, (called optimal hyperplane), to the convex set X at an optimal solution x_0.

(d) Separating hyperplane

A hyperplane $cx = z$ in $\mathbb{R}^n$, is called a separating hyperplane if it separates to convex set S_1 and S_2 such that $S_1 \subseteq \{x : cx \leq z\}$ and $S_2 \subseteq \{x : cx \geq z\}$ we say that it strictly separates S_1 and S_2 if

$$S_1 \subseteq \{x : cx < z\} \text{ and } S_2 \subseteq \{x : cx > z\}.$$



3.3. Some Theorems

Theorem 3.3.1. The set of all feasible solutions

$X = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ is a convex set

$\left(A = [a_{ij}]_{m \times n} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \right)$

Proof: Let $x_1, x_2 \in X$. Then $Ax_1 = b$, $x_1 \geq 0$ and $Ax_2 = b$, $x_2 \geq 0$. Consider $x_3 = \lambda x_1 + (1-\lambda)x_2$, $0 \leq \lambda \leq 1$

Then $x_3 \geq 0$ as $x_1 \geq 0$, $x_2 \geq 0$ and $0 \leq \lambda \leq 1$.

$$\text{Also } Ax_3 = A(\lambda x_1 + (1-\lambda)x_2) = \lambda(Ax_1) + (1-\lambda)(Ax_2)$$
$$= \lambda b + (1-\lambda) b$$
$$= b.$$