SEMESTER-II

LECTURE NOTES ON

REAL SEQUENCES-2ND PART

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REFERENCE BOOK: **REAL ANALYSIS BY S.K.MAPA**

Sequences

$\Rightarrow$ Prove that $\lim_n a^{1/n} = 1$ if $a > 0$ [NOTE :- $\lim_n n^{1/n} = 1$ (a particular case of this result)]

Case 1 $a = 1 \rightarrow a^{1/n} = 1 \therefore \{a^{1/n}\}_n = \{1, 1, \dots\} \rightarrow 1$

Case 2 $a > 1$. $\therefore a^{1/n} > 1$ Let $a^{1/n} = 1 + x_n$ with $x_n > 0$ — (1)

$\therefore a = (1+x_n)^n = 1 + n x_n + {}^nC_2 x_n^2 + \dots + {}^nC_n x_n^n$ (Binomial expansion)

$> 1 + n x_n \; (\because x_n > 0)$

$\therefore \frac{a-1}{n} > x_n = a^{1/n} - 1$

Now $|a^{1/n} - 1| < \left|\frac{a-1}{n}\right| < \varepsilon$ whenever $n \geq n_0$ where $n_0 = \left[\frac{a-1}{\varepsilon}\right] + 1$

Note reqd to prove $\lim_n a^{1/n} = 1$.

Calculating n_0

$\frac{a-1}{n} < \varepsilon$ — (2)

$\Rightarrow n > \frac{a-1}{\varepsilon}$

choose $n_0 = \left[\frac{a-1}{\varepsilon}\right] + 1$

$\therefore \forall n \geq n_0$, eqn (2) holds

$\therefore \lim_n a^{1/n} = 1$.

Case 3 :- $a < 1$ Let $a = \frac{1}{b}$ with $b > 1$.

$\therefore \lim_n a^{1/n} = \lim_n \left(\frac{1}{b}\right)^{1/n} = \lim_n \frac{1}{b^{1/n}} = \frac{\lim_n 1}{\lim_n b^{1/n}} = \frac{1}{1} = 1$

($\because \lim_n b^{1/n} = 1$ by case 2)

Proved

$\Rightarrow$ Prove that $\lim_n x^n = 0$ if $|x| < 1$ (Reference S. K. Mapa)

$\Rightarrow$ Prove that if $\lim_n x_n = 0$ and $a > 0$ then $\lim_n a^{x_n} = 1$

NOTE - These results are useful for doing problems.

• Let $\{x_n\}_n$ be a sequence of +ve numbers such that
$\lim_n \frac{x_{n+1}}{x_n} = l$.

(i) If $0 \le l < 1$ then $\lim_n x_n = 0$ → Null sequence; (ii) If $l > 1$ then $\lim_n x_n = +\infty$ → Properly divergent seq.

Proof: (i) → $0 \le l < 1$
Choose $\varepsilon > 0$ such that $l + \varepsilon < 1$ → Note ε can be chosen in this way such that adding it to $l > 0$ still produces a number < 1

$\because \lim_n \frac{x_{n+1}}{x_n} = l$

$\therefore$ Choose $\varepsilon > 0$, ... K s.t.

$$\left| \frac{x_{n+1}}{x_n} - l \right| < \varepsilon \quad \forall n \ge K$$

$$\Rightarrow l - \varepsilon < \frac{x_{n+1}}{x_n} < \underbrace{l + \varepsilon}_{= r \text{ (say)}} \quad \forall n \ge K$$

$$\therefore \frac{x_{K+1}}{x_K} < r, \quad \frac{x_{K+2}}{x_{K+1}} < r, \quad \frac{x_{K+3}}{x_{K+2}} < r \cdots \frac{x_n}{x_{n-1}} < r$$
(1, 2, 3, ..., $(n-K)$) $\forall n \ge K+1$

Multiplying these inequalities gives

$$\frac{x_n}{x_K} < (r)^{\boxed{n-K}}$$ → no of inequalities appearing above.

$$\Rightarrow 0 < x_n < x_K r^{n-K} = \frac{x_K r^n}{x_K r^K} \quad \forall n \ge K+1 \quad \text{— (i)}$$

Now $\lim_n r^n = 0$ ($\because r < 1$) → This result just discussed)

and $\frac{x_K}{r^K}$ is a fixed positive number.

$\therefore \lim_n x_n = 0$ (By Sandwich theorem)
→ $0 < x_n < \left(\frac{x_K}{r^K}\right) r^n$; $0 \to 0$, $x_n \to 0$, $r^n \to 0$

(ii) Proved in next page

③

(ii) Choose $\varepsilon > 0$ such that $l - \varepsilon > 1$ → Note such a choice is possible.
↳ $l > 1$.

$\because \lim_n \frac{x_{n+1}}{x_n} = l$ ~~definition again~~

∴ For above ε, $\exists$ K - - - - .

$s = l - \varepsilon < \frac{x_{n+1}}{x_n} < l + \varepsilon \quad \forall n \geq K$.
(say)

$\therefore \underbrace{\frac{x_{K+1}}{x_K} > s}_{1^{st}}, \underbrace{\frac{x_{K+2}}{x_{K+1}} > s}_{2^{nd}} \cdots \underbrace{\frac{x_n}{x_{n-1}} > s}_{n-K^{th} \text{ inequality}} \quad \forall n \geq K+1$

multiplying . .

$\frac{x_n}{x_K} > s^{n-K} \Rightarrow x_n > \frac{x_K}{s^K} \cdot s^n \quad \forall n \geq K+1$

Now as $s > 1 \quad \therefore \lim_n s^n = +\infty$.

Also $\frac{x_K}{s^K}$ is a fixed +ve number.

Proved.

$\therefore \lim_n x_n = +\infty$

<u>Note</u> :- ① This result gives nature of convergence of original sequence $\{x_n\}_n$ using range of limit of sequence formed by ratio of consecutive elements. $\left\{\frac{x_{n+1}}{x_n}\right\}$.

② = Imp No definite conclusion if $l = 1$. → 2 examples required. (for which $\lim_n x_n = 0$ hold; for which $\lim_n x_n \neq 0$)

(i) $x_n = \frac{n+1}{n} \quad \therefore \frac{x_{n+1}}{x_n} = \frac{n+2}{n+1} \cdot \frac{n+1}{n} = 1 + \frac{2}{n}$

$\therefore \lim_n \left(\frac{x_{n+1}}{x_n}\right) = 1$ and $\lim_n x_n = \lim_n \left(1 + \frac{1}{n}\right) = 1 \underline{\neq 0}$

But for (ii) $x_n = 1/n$; $\lim_n x_n \underline{= 0}$ and $\lim_n \frac{x_{n+1}}{x_n} = 1$

→ Prove the following (Reference S.K. Mapa Book)

Let $\{x_n\}_n$ be a sequence of positive numbers such that $\lim_n x_n^{1/n} = l$.

(i) If $0 \le l < 1$ then $\lim_n x_n = 0$ (Null sequence)

(ii) If $l > 1$ then $\lim_n x_n = +\infty$ (Properly divergent)

<u>Note</u> :- ① Here also convergence of original sequence $\{x_n\}_n$ can be studied (whether null or properly divergent sequence) by nature of limit of a <u>new</u> sequence constructed from <u>original seq</u> $\{x_n\}_n$.

↳ $\{x_n^{1/n}\}_n$.

② Here also no conclusion for $l = 1$ → Need to provide 2 examples. Consider the following two

(i) $\{x_n\}_n = \left\{\frac{n+1}{n}\right\}_n$. $\lim_n x_n = 1$; $\underbrace{\lim_n x_n^{1/n} = 1}$ $\left(1+\frac{1}{n}\right)^{1/n}$ ↳ (discussed later)

(ii) $\{x_n\}_n = \left\{\frac{1}{n}\right\}_n$; $\lim_n x_n = 0$; $\lim_n x_n^{1/n} = 1$

↓

$\lim_n n^{-1/n}$

$= \frac{1}{\lim_n n^{1/n}} = \frac{1}{1} = 1$

↓

discussed before.

Note $\boxed{\begin{array}{l}\lim_n n^{1/n} = 1 \\ \lim_n \left(1+\frac{1}{n}\right)^{1/n} = 1\end{array}}$

↳ follows from result $\lim_n a^{1/n} = 1$ where $a > 0$.

→ Let $\{x_n\}_n$ be defined by $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)\ \forall\, n \geq 1$ and $0 < x_1 < x_2$. Prove that $\{x_n\}_n \longrightarrow \frac{x_1 + 2x_2}{3}$.

Pf $0 < x_2 - x_1 = \left(\frac{1}{2}\right)^0 (x_2 - x_1)$

Now $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$

$$x_3 - x_2 = \frac{1}{2}(x_2 + x_1) - x_2 \quad \text{(Using given rel}^n\text{)}$$
$$= \left(-\frac{1}{2}\right)^1 (x_2 - x_1)$$

$$x_4 - x_3 = \frac{1}{2}(x_3 + x_2) - x_3$$
$$= -\frac{1}{2}(x_2 - x_3)$$
$$= \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x_2 - x_1)$$
$$= \left(-\frac{1}{2}\right)^2 (x_2 - x_1)$$

Continuing similarly

$$x_n - x_{n-1} = \left(-\frac{1}{2}\right)^{(n-1)-1} (x_2 - x_1)$$

Adding all these relations we;

Geometric series common ratio $(-1/2) < 1$ and $n-1 \to$ no of terms

$\cancel{x_2 - x_1 + x_3 - x_2 + x_4 - x_3 +} \cdots + x_n - x_{n-1}$

$\cancel{x_2} - x_1 + \cancel{x_3} - \cancel{x_2} + \cancel{x_4} - \cancel{x_3} + \cdots + \cancel{x_{n-1}} - \cancel{x_{n-2}} + x_n - \cancel{x_{n-1}}$

$$\Rightarrow x_n - x_1 = (x_2 - x_1)\left[1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2 + \cdots + \left(-\frac{1}{2}\right)^{n-2}\right]$$
$$= (x_2 - x_1)\left[\frac{1 - \left(-\frac{1}{2}\right)^{n-1}}{1 - \left(-\frac{1}{2}\right)}\right]$$

$$\Rightarrow x_n - x_1 = \frac{2(x_2 - x_1)}{3}\left[1 - \left(-\frac{1}{2}\right)^{n-1}\right]$$

∴ Taking $\lim_n (x_n - x_1) = \frac{2}{3}(x_2 - x_1) \Rightarrow \lim_n x_n =$ Required value