SEMESTER-II

LECTURE NOTES ON

Sequence

5TH PART

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REFERENCE BOOK: **REAL ANALYSIS**

BY

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Subsequential Limit

Let $\{x_n\}_n$ be a real sequence. $l \in \mathbb{R}$ is said to be a subsequential limit of $\{x_n\}_n$ if $\exists$ a subsequence $\{x_{r_n}\}_n$ of $\{x_n\}_n$ that converges to l

eg:- $\{(-1)^n\}_n \longrightarrow$ Has 2 subsequential limits -1 and 1

$\downarrow$

(Justification:- Consider subseq $\{(-1)^{2n}\}_n$.

$$\underset{n}{Lt}\ (-1)^{2n} = 1$$

$\therefore \{(-1)^{2n}\}_n \rightarrow 1 \quad \therefore 1 \rightarrow$ subsequential limit.

Similarly consider $\{(-1)^{2n-1}\}_n$ to prove $-1 \rightarrow$ subsequential limit.

$\Rightarrow$ We now try to sort out criterion to detect whether a given real number say l is a subsequential limit of a given sequence $\{x_n\}_n$.

$\Rightarrow$ $l \in \mathbb{R}$ is a subsequential limit of a given sequence $\{x_n\}_n$ iff $\forall$ nbd. of l contains infinitely many elements of $\{x_n\}_n$

Proof :→ ~~Condition Necessary~~ Let l be a subsequential limit of $\{x_n\}_n$

$\therefore$ By def^n, $\exists \{x_{r_n}\}_n$ such that $\underset{n}{Lt}\, x_{r_n} = l$.

$\therefore$ choose $\varepsilon > 0$, $\exists k \in \mathbb{N}$ (k depending on ε) such that

$l - \varepsilon < x_{r_n} < l + \varepsilon \quad \forall n \geq k$.

i.e. $x_{r_n} \in N_\varepsilon(l) \; \forall n \geq k$

$\because$ ε is arbitrary.

$\therefore$ Every nbd. of l contains infinitely many elements of $\{x_n\}_n$ Proved

Condition sufficient :→ Let $\{x_n\}_n$ be the sequence such that every nbd of l contains ∞ many elements of $\{x_n\}_n$.

$\therefore$ for any pre-assigned $\varepsilon > 0$ $N_\varepsilon(l)$ contains ∞ many elements of $\{x_n\}_n$

Let $\varepsilon = 1$. Then $x_n \in N_1(l)$ for ∞ many n

Define $S_1 = \{n \in \mathbb{N} \mid x_n \in N_1(l)\}$

$\therefore$ $S_1 \subset \mathbb{N}$ and is infinite

$\therefore$ By well-ordering property of $\mathbb{N}$, S_1 has a least element say x_{r_1}.

$\therefore$ $l - 1 < x_{r_1} < l + 1$ — (1)

Let $\varepsilon = 1/2$, then $x_n \in N_{1/2}(l)$ for infinitely many n

Define $S_2 = \{n \in \mathbb{N} \mid x_n \in N_{1/2}(l)$

$\therefore$ $S_2 \subseteq \mathbb{N}$ and is infinite.

$\therefore$ By well ordering property of $\mathbb{N}$, S_2 has a least element say x_{r_2} such that $r_2 > r_1$

$\therefore$ $l - 1/2 < x_{r_2} < l + 1/2$ — (2)

Continuing similarly we get $\{r_1, r_2, \dots\} \in \mathbb{N}$ and $\{x_{r_1}, x_{r_2}, \dots\}$ such that

$r_1 \le r_2 \le \cdots$ and

$L - \frac{1}{k} < x_{r_k} < L + \frac{1}{k} \quad \forall\, k \in \mathbb{N}$ — (2)

~~subsequence~~

$\because\, r_1 \le r_2 \le \cdots$ with $r_i \in \mathbb{N}$.

$\therefore \{x_{r_k}\}_k$ is a subsequence of $\{x_n\}_n$

Again eq^n (2) implies.

$\displaystyle \lim_{n\to\infty} x_{r_n} = L$ (By Sandwich theorem)

$\therefore L \Rightarrow$ subsequential limit of $\{x_n\}_n$

Proved.

<u>Note</u> :- (1) Limit of a sequence $\{x_n\}_n$ is obviously a subsequential limit of $\{x_n\}_n$

(2) Subsequential limits of $\{x_n\}_n \rightarrow$ Limit Points of Range set of $\{x_n\}_n \rightarrow$ Easy To Argue Using definition of Limit Point.

(Try It).

Peaks of $\{x_n\}_n$

Defn

An element x_K of $\{x_n\}_n$ is said to be a peak of $\{x_n\}_n$ if $x_K \geq x_n \ \forall n > K$

eg:→ $\{(-1)^n\}_n$ → Peak of this sequence is 1

as $x_{2n} = 1 \ \forall n \in \mathbb{N}$

∴ $x_2 = 1$.

and $x_n \geq x_2 \ \forall n > 2$.

Note :- $\{x_n\}_n$
- → Has No, Peak I
- → Has Peak II
  - Finite No. of Peaks III
  - Infinite No of Peaks IV

Egs (I) → $\{n(-1)^n\}_n = \{-1, 2, -3, 4, -5, \ldots\}_n$

∄ any element $x_K \in \{n(-1)^n\}_n$ such that $x_K \geq x_n \ \forall n > K$.

II → $\{(-1)^n\}_n$ → 1 is the peak

III $\{(-1)^n\}_n$, $\left\{\sin \frac{n\pi}{4}\right\}_n$

IV → $\left\{\frac{1}{n}\right\}_n$ → Every element is a peak. (∞ no of peaks)

Justification that $\{\frac{1}{n}\}_n$ has ∞ no of peaks

$x_1 = 1$.

$\therefore x_n < x_1 \quad \forall n > 1$

Again $x_2 = \frac{1}{2} \quad \therefore x_n < \frac{1}{2} \quad \forall n > 2$

So consider x_k.

$\therefore x_n < x_k = \frac{1}{k} \quad \forall n > k$.

and this is true $\forall k \in \mathbb{N}$.

$\therefore$ Every element of $\{x_n\}_n$ is a peak of $\{x_n\}_n$. So $\{x_n\}_n$ has ∞ no of peaks

NOTE :- Concept of peaks of $\{x_n\}_n$ used to prove a very significant result :

Important

Every sequence $\{x_n\}_n$ has a $\boxed{\text{monotone}}$ $\{x_{r_n}\}_n$

$\{x_n\}_n$ can be of any nature by def^n. But it will always have a subsequence.

Not only any subsequence but $\{x_n\}_n$ will have a monotone subsequence.

⇒ Every real sequence $\{x_n\}_n$ has a monotone subsequence $\{x_{r_n}\}_n$

Proof :→ Case 1 :→ $\{x_n\}_n$ has ∞ no of peaks.

Let $x_{r_1}, x_{r_2}, x_{r_3} \ldots$ be the peaks such that

$x_{r_1}, x_{r_2}, x_{r_3} \ldots$ be the $1^{st}, 2^{nd}, 3^{rd}, \ldots$ peaks.

(i.e. $r_1 \le r_2 \le r_3 \le \ldots$)

∴ By def^n of peaks

$x_{r_1} \geq x_{r_2} \geq x_{r_3} \geq \ldots$,

Consider coll^n of the peaks $\{x_{r_n}\}_n$.

Clearly $\{x_{r_n}\}_n$ i.e coll^n of peaks forms a monotone ↓ subseq of $\{x_n\}_n$

Case 2 :→ $\{x_n\}_n$ has either no peak or finite no of peaks.

Let $x_{r_1}, x_{r_2}, \ldots x_{r_m}$ → peaks of $\{x_n\}_n$

Let $s_1 = r_m + 1$, ∴ x_{s_1} → not a peak

∵ x_{s_1} is not peak.

∴ $\exists\, s_2 \in \mathbb{N}$ s.t $x_{s_2} > x_{s_1}$.

x_{s_2} → Not a peak.

∵ x_{s_2} not a peak ∴ $\exists\, s_3 \in \mathbb{N}$ s.t $x_{s_3} > x_{s_2}$.

Proceeding in this way, $s_1, s_2, s_3 \ldots \in \mathbb{N}$ are obtained

such that $s_1 < s_2 < s_3$ and $u_{s_1} < u_{s_2} < u_{s_3} < \ldots$

$\therefore \{u_{s_n}\}_n$ is a monotone $\uparrow$ subsequence of

Proved.

<u>NOTE</u> (<u>From the proof</u>)

If ∞ no of peaks exist then monotone $\downarrow$ subseq present definite

If finite number or no peak present then monotone $\uparrow$ subsequence present definitely.

Now, we know that every seq has a monotone subseq.

But if the seq is now bdd then it has a convergent subseq. (Boltzano-Weierstrass theorem)

Proof :→ Let $\{x_n\}_n$ be a bdd sequence.

∴ ∃ a closed bdd interval $I = [a, b]$ (say)

such that $x_n \in I \ \forall n \in \mathbb{N}$.

Let $c = \frac{a+b}{2}$ and let $I' = [a, c]$, $I'' = [c, b]$

∴ At least one of I' or I'' must contain ∞ many elements of $\{x_n\}_n$ ($\because x_n \in I \ \forall n \in \mathbb{N}$)

∴ without loss of any generality, let I' contains ∞ many elements of $\{x_n\}_n$

Let $I_1 = I' = [a, c] = [a_1, b_1]$ (say)

Let $c_1 = \frac{a_1 + b_1}{2}$. Let $I_1' = [a_1, c_1]$, $I_1'' = [c_1, b_1]$

Then again at least one of I_1', I_1'' must contain ∞ many elements of $\{x_n\}_n$

without loss of any generality

let I_1' contains ∞ many elements of $\{x_n\}_n$

Let $I_2 = I_1' = [a_2, b_2]$ (say)

Continuing in this way a sequence of closed & bdd intervals $\{I_n\}_n$ are obtained

such that :-

(i) $I_{n+1} \subset I_n \ \forall n \in \mathbb{N}$.

(ii) $|I_n| = \frac{1}{2^n}(b-a)$ & hence $\lim_{n\to\infty} |I_n| = 0$.

(iii) each I_n contains ∞ many elements of $\{x_n\}_n$.

$\therefore$ By Cantor's Theorem on nested interval

$\exists$ unique α such that $\alpha \in \bigcap_{n=1}^{\infty} I_n$.

To prove α is a subsequential limit of $\{x_n\}_n$

choose $\varepsilon > 0$. $\therefore \exists\, k_\varepsilon \in \mathbb{N}$ such that

$$0 < \frac{b-a}{2^{k_\varepsilon}} < \varepsilon \quad \Rightarrow |I_{k_\varepsilon}| < \varepsilon$$

Now $\alpha \in I_K$

and I_K is entirely contained in $(\alpha-\varepsilon, \alpha+\varepsilon)$
$(\because |(\alpha-\varepsilon, \alpha+\varepsilon)| = 2\varepsilon > \varepsilon)$

Again I_K contains ∞ many elements of $\{x_n\}_n$,

$\therefore$ $(\alpha-\varepsilon, \alpha+\varepsilon)$ contains ∞ many elements of $\{x_n\}_n$.

$\therefore$ $N_\varepsilon(\alpha)$ contains ∞ many elements of $\{x_n\}_n$

$\therefore$ $\alpha \rightarrow$ subsequential limit of $\{x_n\}_n$

($\because$ Every subsequential limit of a seq contains ∞ many elements of the sequence)

$\therefore$ $\exists$ a subseq $\{x_{r_n}\}_n$ of $\{x_n\}_n$ that converges to α.

$\therefore$ $\{x_{r_n}\}_n$ is a converging subseq of $\{x_n\}_n$.

$\therefore$ $\{x_n\}_n$ has a converging subsequence

Proved