**SEMESTER-II**

**LECTURE NOTES ON**

**Sequence**

**$6^{TH}$ PART**

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REFERENCE BOOK: **REAL ANALYSIS**

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## Upper And Lower Subsequential limit

Let $\{x_n\}_n \to$ bdd. By Boltzano Weierstrass Theorem, $\exists$ a convergent subseq of $\{x_n\}_n$.
$\therefore \exists$ subsequential limit of $\{x_n\}_n$.
Now $\{x_n\}_n$ is bounded $\therefore$ so set $S$ of all subsequential limits of $\{x_n\}_n$ is a bounded set.

<u>Case 1</u> :$\to$ $S$ is a finite set. Then $S$ has a greatest element.

<u>Case 2</u> :- $S$ is an infinite set. $\because$ Bdd so $S$ has a least upper bound.

Let $\sup(S) = u^*$.

$\therefore \exists$ element of $S$ say $l_1 > u^* - 1$.
But $l_1 \to$ subsequential limit of $\{x_n\}_n$.
$\therefore \exists r_1$ such that $x_{r_1} > u^* - 1$.
Again $\exists\ l_2 \in S$ such that $l_2 > u^* - \frac{1}{2}$.
$\therefore \exists r_2 \in \mathbb{N}$, $r_2 > r_1$ such that
$x_{r_2} > u^* - \frac{1}{2}$

$\therefore$ Continuing, we get

a strictly $\uparrow$ seq of natural number

$\{r_1, r_2, \dots\}$ such that $x_{r_n} > u^* - \frac{1}{n}$ $\forall n \in \mathbb{N}$

Now for any $\varepsilon > 0$, $\exists k \in \mathbb{N}$ such that

$0 < \frac{1}{n} < \varepsilon \quad \forall n \geq k$.

$\therefore \quad |x_{r_n} - u^*| < \frac{1}{n} < \varepsilon \quad \forall n \geq k$.

$\because \varepsilon \to$ arbitrary

$\therefore \quad u^* \to$ subsequential limit of $\{x_n\}_n$.

But $u^* = \sup(S)$

$\therefore u^* \to$ greatest subsequential limit of $\{x_n\}_n$.

Similarly bdd $\{x_n\}_n$ has a least subsequential limit (Prove It).

Defn Let $\{x_n\}_n$ be a bounded sequence of real numbers. The greatest subsequential limit of $\{x_n\}_n$ is said to be the upper limit of $\{x_n\}_n$ or limit superior of $\{x_n\}_n$.

Denoted by $\overline{\underset{n}{Lt}}\, x_n$ or $\underset{n}{Lt} \sup x_n$.

Similarly, the least subsequential limit of $\{x_n\}_n$ is said to be the lower limit of $\{x_n\}_n$ or limit inferior of $\{x_n\}_n$.

Denoted by $\underline{\underset{n}{Lt}}\, x_n$ or $\underset{n}{Lt} \inf x_n$.

-> If $\{x_n\}_n$ is unbdd above, $\overline{Lt}\, x_n = \infty$

-> If $\{x_n\}_n$ is unbdd below, $\underline{\underset{n}{Lt}}\, x_n = -\infty$.

Conditions for $u^*$ to be limit superior of $\{x_n\}$:

$\forall \varepsilon > 0$ the following 2 conditions must hold simultaneously.

i) $x_n > u^* - \varepsilon$ for $\infty$ many $n$

ii) $\exists k \in \mathbb{N}$ such that $x_n < u^* + \varepsilon \ \forall n \geq k$.

(Justify second criteria(ii)).

Conditions for $u_* = \underline{Lt}_n\, x_n$

For all $\epsilon > 0$, , followings must hold simultaneously

(i) $x_n < u_* + \epsilon$ for $\infty$ many $n$.

(ii) $\exists\, k \in \mathbb{N}$ s.t $x_n > u_* - \epsilon \;\; \forall\, n \geq k$.

Alternatively $\underline{Lt}_n\, x_n$, $\overline{Lt}_n\, x_n$ can be determined as follows:-

Case 1 $\{x_n\}_n$ bdd above.

Let $M_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}$

$m_n = \inf\{x_n, x_{n+1}, x_{n+2}, \dots\}$

$\therefore M_1 = \sup\{x_1, x_2, x_3, x_4, \dots\}$

$M_2 = \sup\{x_2, x_3, x_4 \dots\}$

$M_3 = \sup\{x_3, x_4, x_5, \dots\}$

$\vdots$

$\therefore M_1 \geq M_2 \geq M_3 \geq \dots$

$\therefore \{M_n\}_n$ is $\downarrow$. and $Lt_n\, M_n = \overline{Lt}_n\, x_n = u^*$

SubCases :→ (1) $\{x_n\}_n$ is unbdd below.

Then $Lt_n\, M_n = -\infty$. i.e $u^* = -\infty$

$\therefore \{x_n\}_n$ bdd above & unbdd below then $u^* = -\infty$.

③ $\{x_n\}_n$ is bdd below. Then $\underset{n}{Lt}\, M_n = u^*$ where $u^*$ is the greatest lower bound of $\{M_n\}_n$

$$u^* = \inf_n (M_n)$$

Case 2 $\{x_n\}_n$ unbdd above.

Then clearly, $M_1 = M_2 = M_3 = \cdots = +\infty$.

$\therefore \overline{\underset{n}{Lt}}\, x_n = +\infty$.

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Now determining $\underline{\underset{n}{Lt}}\, x_n = u_*$.

Case 1 $\{x_n\}_n$ bdd below.

$m_1 = \inf\{x_1, x_2, x_3, x_4 \cdots\}$

$m_2 = \inf\{x_2, x_3, x_4, \cdots\}$

$\vdots$

$\therefore m_1 \leq m_2 \leq m_3 \leq \cdots$

$\therefore \{m_n\}_n$ is $\uparrow$. and $\underset{n}{Lt}\, m_n = \underline{\underset{n}{Lt}}\, x_n = u_*$

Sub cases ① $\{x_n\}_n$ is unbdd above.

Then $\underset{n}{Lt}\, m_n = +\infty$ i.e. $u_* = +\infty$.

$\therefore \{x_n\}_n$ bdd below & unbdd above then $u_* = \infty$

Subcase 2 :- $\{x_n\}_n$ is bdd above

Then $\underset{n}{Lt}\, m_n = u_*$ where $u_*$ is the supremum of $\{m_n\}_n$. i.e $u_* = \sup\limits_{n} (m_n)$

Case 2 :- $\{x_n\}_n$ is unbdd below.

Then clearly, $m_1 = m_2 = m_3 = \cdots = -\infty$.

$\therefore \underline{\underset{n}{Lt}}\, x_n = -\infty$.

Let $x_n = (-1)^n \left(1+\frac{1}{n}\right)$, $n \geq 1$. Determine

$\overline{Lt}\, x_n$, $\underline{Lt}\, x_n$.

$$\{x_n\}_n = \left\{ -2,\ 1+\frac{1}{2},\ -\left(1+\frac{1}{3}\right),\ \left(1+\frac{1}{4}\right),\ -\left(1+\frac{1}{5}\right),\ \left(1+\frac{1}{6}\right), \cdots \right.$$

Let $M_n = \sup\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \cdots\}$

$M_1 = 1+\frac{1}{2}$ ; $M_2 = 1+\frac{1}{4}$, $\ldots M_n = \left(1+\frac{1}{2n}\right)$

$$\therefore \underset{n}{Lt}\, M_n = \underset{n}{Lt}\left(1+\frac{1}{2n}\right) = 1 = \overline{\underset{n}{Lt}}\, x_n$$

Similarly, let $m_n = \inf\{x_n, x_{n+1}, x_{n+2}, \cdots\}$

$\therefore m_1 = -2$, $m_2 = -\left(1+\frac{1}{3}\right)$, $m_3 = -\left(1+\frac{1}{5}\right)$

$\cdots m_n = -\left(1+\frac{1}{2n-1}\right)$

$$\therefore \underset{n}{Lt}\, m_n = -1 = \underline{Lt}\, x_n$$

$\therefore \overline{\underset{n}{Lt}}\, x_n = 1$, $\underset{\underline{n}}{Lt}\, x_n = -1$.

i) $\{x_n\}_n = \{(-1)^n n\}$.
$= \{-1, 2, -3, 4, -5, \cdots\}$

$M_n = \sup\{x_n, x_{n+1}, \cdots\}$

$m_n = \inf\{x_n, x_{n+1}, \cdots\}$

$M_1 = M_2 = M_3 = \cdots = +\infty$

$m_1 = m_2 = m_3 = \cdots = -\infty$

$\therefore \underset{n}{Lt}\, M_n = +\infty \quad \therefore \overline{Lt}_n\, x_n = +\infty$

and $\underset{n}{Lt}\, m_n = -\infty \quad \therefore \underline{Lt}_n\, x_n = -\infty$.

i) $\{x_n\}_n = \{-n^2\}_n = \{-1^2, -2^2, -3^2, -4^2, -5^2, \cdots\}$

$M_n = \sup\{x_n, x_{n+1}, \cdots\}$

$M_1 = -1^2, M_2 = -2^2, \cdots, M_n = -n^2$

$\therefore \underset{n}{Lt}\, M_n = -\infty \quad \therefore \overline{Lt}\, x_n = -\infty$.

$m_n = \inf\{x_n, x_{n+1}, x_{n+2}, \cdots\}$

$m_1 = m_2 = \cdots = m_n = \cdots = -\infty$.

$\therefore \underset{n}{Lt}\, m_n = -\infty \quad \therefore \underline{Lt}\, x_n = -\infty$.