SEMESTER-II

LECTURE NOTES ON

Sequence

7TH PART

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REFERENCE BOOK: **REAL ANALYSIS**

BY

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Subsequential Limit

$\Rightarrow$ A bdd seq $\{x_n\}_n$ is Convg iff $\overline{Lt}_n\, x_n = \underline{Lt}_n\, x_n$.

Proof (Necessary Part) :→ Let $\{x_n\}_n$ be convg.

Let $Lt_n\, x_n = l$.

$\because \{x_n\}_n$ is convg to l so every subseq of $\{x_n\}_n$ also converges to l.

$\therefore [l =$ greatest as well as smallest subsequential limit$]$ → Try to prove this statement

$\therefore l = \overline{Lt}_n\, x_n = \underline{Lt}_n\, x_n$ (Proved)

Sufficient Part :- Let $\overline{Lt}_n\, x_n = \underline{Lt}_n\, x_n = m$ (say).

$\because \overline{Lt}_n\, x_n = m$.

$\therefore$ for $\varepsilon > 0$, $\exists\, k_1 \in \mathbb{N}$ (k_1 depending on ε) s.t

$x_n < m + \varepsilon \quad \forall n \geq k_1$

Again as $\underline{Lt}_n\, x_n = m$.

$\therefore$ for $\varepsilon > 0$, $\exists\, k_2 \in \mathbb{N}$ - - - - -

$x_n > m - \varepsilon \quad \forall n \geq k_2$.

Let $k = \max\{k_1, k_2\}$.

$\therefore \forall n \geq k$, $|x_n - m| < \varepsilon \Rightarrow Lt_n\, x_n = m$

$\Rightarrow \{x_n\}_n$ converges

Let $\{x_n\}_n$ and $\{y_n\}_n$ be bdd sequences. Then

(i) $\underline{\mathop{Lt}_n}\, x_n + \underline{\mathop{Lt}_n}\, y_n \le \underline{\mathop{Lt}_n}\,(x_n + y_n)$

(ii) $\overline{\mathop{Lt}_n}\, x_n + \overline{\mathop{Lt}_n}\, y_n \ge \overline{\mathop{Lt}_n}\,(x_n + y_n)$

Proof (1) $\because$ $\{x_n\}_n$ & $\{y_n\}_n$ both bdd, so

$\{x_n + y_n\}_n$ is also bdd

~~$\therefore \mathop{Lt}_n (x_n+y_n)$, $\mathop{Lt}_n x_n$, $\mathop{Lt}_n y_n$ all exists finitely~~

~~Let $\mathop{Lt}_n$~~

$\therefore \underline{\mathop{Lt}_n}\, x_n = l_1$ (say), $\underline{\mathop{Lt}_n}\, y_n = l_2$ (say) & $\mathop{Lt}_n\, x_n + y_n = l_3$ (say)

where l_1, l_2, l_3 all exist finitely.

Now As $\mathop{Lt}_n x_n = l_1$.

$\therefore$ For $\varepsilon > 0$, $\exists\, k_1 \in \mathbb{N}$ (k_1 depending on ε) such that

$l_1 - \varepsilon/2 < x_n \quad \forall\, n \ge k_1$

Again as $\underset{n}{\underline{Lt}}\, y_n = l_2$.

$\therefore$ for $\varepsilon > 0$, $\exists\, k_2 \in \mathbb{N}$ - - - -

$$l_2 - \varepsilon_2 < y_n \quad \forall\, n \geq k_2$$

Let $k = \max\{k_1, k_2\}$

$\therefore$ for $\varepsilon \geq 0$

$$l_1 + l_2 - \varepsilon < x_n + y_n \quad \forall\, n \geq k.$$

$\therefore$ NO subsequential limit of $\{x_n + y_n\}_n$ can be smaller than $l_1 + l_2 - \varepsilon$.

Now as $\varepsilon > 0$ is arbitrary.

$\therefore$ NO subseq limit of $\{x_n + y_n\}_n$ can be smaller than $l_1 + l_2$.

$\therefore$ l_3 cannot be smaller than $l_1 + l_2$.

$\therefore$ $l_3 \geq l_1 + l_2$.

$\therefore$ $\underset{n}{\underline{Lt}}\, x_n + \underset{n}{\underline{Lt}}\, y_n \leq \underset{n}{\underline{Lt}}\, (x_n + y_n)$ Proved.

(ii) Try on your own.

→ verify the above result for $\{x_n\}_n = \{\sin n\pi/2\}$ and $\{y_n\}_n = \{\cos n\pi/2\}_n$.

Note :- 1) The above result states relation b/w 2 subsequential limits addition with that of subsequential limit of the new sequence $\{x_n+y_n\}$ created by adding (termwise) two ~~...~~ bdd sequences $\{x_n\}_n$, $\{y_n\}_n$.

2) For above relⁿ to hold, there is no need for the given sequences $\{x_n\}_n$, $\{y_n\}_n$ to be convergent (~~...~~ Problem to be verified above gives one such example). ~~...~~ Boundedness (from below & above) is enough.

3) Boundedness (from both sides) is necessary for strict inequality. otherwise '=' occurs only (Also boundedness of both sequences reqd).

Say $\{x_n\}_n$, $\{y_n\}_n$ both unbdd above

then $\{x_n+y_n\}_n$ unbdd above.

$$\therefore \overline{Lt}_n\, x_n = \overline{Lt}_n\, y_n = +\infty = \overline{Lt}_n\,(x_n+y_n)$$

$$\therefore \overline{Lt}_n(x_n) + \overline{Lt}_n\, y_n = \overline{Lt}_n\,(x_n+y_n)$$

even if say only $\{x_n\}_n$ is unbdd above but $\{y_n\}_n$ is bdd. then also.

$\{x_n+y_n\}_n$ unbdd above.

$$\therefore \overline{Lt}_n\, x_n = +\infty, \quad \overline{Lt}_n\, y_n = l \text{ (say)}$$

$\overline{Lt}_n\,(x_n+y_n) = +\infty$ clearly $\overline{Lt}_n\, x_n + \overline{Lt}_n\, y_n = +\infty$

$\therefore$ Again only equality holds.

Similar is the case for unbdd below sequences.

Cauchy Criterion

Consider $\{x_n\}_n$ to be a convg seq. Let $\underset{n}{Lt}\, x_n = l$.

$\therefore$ By defn of convg, we know that for any pre-assigned $\varepsilon > 0$, we always get a suitable natural number (k, say) such that once k is crossed (i.e. for any natural number $\geq k$), corresponding sequence elements (i.e $x_n \;\forall n \geq k$) lie very close to l (guided by ε which can be made arbitrarily small).

$\downarrow$

$$x_n \in (l - \varepsilon, l + \varepsilon) \;\forall n \geq k.$$

Now if this happens, then obviously distance b/w the sequence elements x_n', x_n'' (say) also becomes $< \varepsilon$ ($\forall n', n'' \geq k$).

So all the seq elements come arbitrarily close to each other once k is crossed.

So $|x_n' - x_n''| < \varepsilon \;\forall n \geq k$ [Roughly this gives Cauchy Criterion of convergence]

Cauchy's General Principle of Convergence (Cauchy criterion of convergence)

A necessary and sufficient condition for the convergence of a seq $\{x_n\}_n$ is that for a pre-assigned $\varepsilon > 0$, $\exists\ k \in \mathbb{N}$ (k depending on ε) such that.

$$|x_{n+p} - x_n| < \varepsilon \quad \forall\ n \geq k \text{ and } p = 1, 2, 3, \ldots$$

Proof: → (Necessary Part): Let $\{x_n\}_n$ be convergent to l.

$\therefore$ Choose $\varepsilon > 0$, $\exists\ k \in \mathbb{N}$ ....

$$|x_n - l| < \varepsilon/2 \quad \forall\ n \geq k.$$

$$\therefore \quad |x_{n+p} - l| < \varepsilon/2 \quad \forall\ p = 1, 2, 3 \ldots \ \& \ \forall\ n \geq k.$$

Now $|x_{n+p} - x_n| = |(x_{n+p} - l) - (x_n - l)|$

$$\leq |x_{n+p} - l| + |x_n - l| < \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon \quad \forall\ n \geq k \text{ and } p = 1, 2, 3, \ldots$$

$\therefore \quad |x_{n+p} - x_n| < \varepsilon \quad \forall\ n \geq k$ and $p = 1, 2, 3 \ldots$

(Proved)

<u>Condition Sufficient</u>:- To prove $\{x_n\}_n$ is convergent.

By given condition, for $\varepsilon = 1$, $\exists\, k_1 \in \mathbb{N}$ such that

$$|x_{n+p} - x_n| < 1 \quad \forall\, n \geq k_1 \ \& \ p = 1, 2, 3 \ldots$$

$$\therefore |x_{k+p} - x_k| < 1 \quad \forall\, p = 1, 2, 3 \ldots \quad \text{—— (1)}$$

Let $B = \max\{x_1, x_2, \ldots x_k \pm 1\}$ $b = \min\{x_1, x_2, \ldots, x_k - 1\}$

b

By (1) $-1 + x_k < x_{k+p} < 1 + x_k \quad \forall\, p = 1, 2, 3 \ldots$

$\therefore$ $b \leq x_n \leq B \quad \forall\, n \in \mathbb{N}$.

$\therefore \{x_n\}_n$ is a bdd sequence.

$\therefore$ By Boltzano Weierstrass Theorem, $\{x_n\}_n$ has a converging subsequence. Let l be the limit of the subseq. Then l is a subsequential limit of $\{x_n\}_n$

Let $\varepsilon > 0$, By given cond^n. $\exists\, k_2 \in \mathbb{N}$ s.t. (k_2 depending on ε)

$$|x_{n+p} - x_n| < \varepsilon/3 \quad \forall n \geq k_2,\ p = 1, 2, \ldots \quad \text{—} (2)$$

For $n = k$, $\quad |x_{k+p} - x_k| < \varepsilon/3 \quad$ — (3)

Now as l is a subsequential limit of $\{x_n\}_n$, each $N_\varepsilon(l)$ must contain infinitely many elements of $\{x_n\}_n$.

$\therefore \exists\, k_3 \in \mathbb{N}$ such that $k_3 > k_2$ and

$$|x_{k_3} - x_k| < \varepsilon/3 \quad \text{—} \quad (4)$$

Now $|x_{k+p} - l| = |x_{k+p} - x_{k_3} + x_{k_3} - x_k + x_k - l|$

$$\leq |x_{k+p} - x_{k_3}| + |x_{k_3} - x_k| + |x_k - l|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \quad (\text{By } (2), (4))$$

$$= \varepsilon \quad \forall p = 1, 2, 3 \ldots$$

$\therefore |x_n - l| < \varepsilon \quad \forall n \geq \underbrace{k+1}_{\text{smallest}}$ $\quad (\because p = 1 \rightarrow |x_{k+1} - l| < \varepsilon$, $p = 2 \rightarrow |x_{k+2} - l| < \varepsilon \ldots$

$\because \varepsilon$ is arbitrary. $\therefore \{x_n\}_n \to l$. Proved.

<u>Note</u> :- <u>Advantage of Cauchy Criterion</u> :→ No need to have knowledge about limit of a converging seq $\{x_n\}_n$ (In the criteria l does not exist ~~as~~ $|x_{n+p} - x_n| < \varepsilon$)

<u>Cauchy Sequence</u> :- $\{x_n\}_n$ is said to be Cauchy sequence if for a pre-assigned $\varepsilon > 0$, $\exists k \in \mathbb{N}$ (k depending on ε) such that

$$\boxed{|x_m - x_n| < \varepsilon \quad \forall m, n \geq k}$$

(Note :- Replace m by n+p with $p = 1, 2, 3, \ldots$ so as to obtain

$$|x_{n+p} - x_n| < \varepsilon \quad \forall n \geq k, \; p = 1, 2, 3 \ldots$$