SEMESTER-II

LECTURE NOTES ON

REAL SEQUENCES-1^{st} PART

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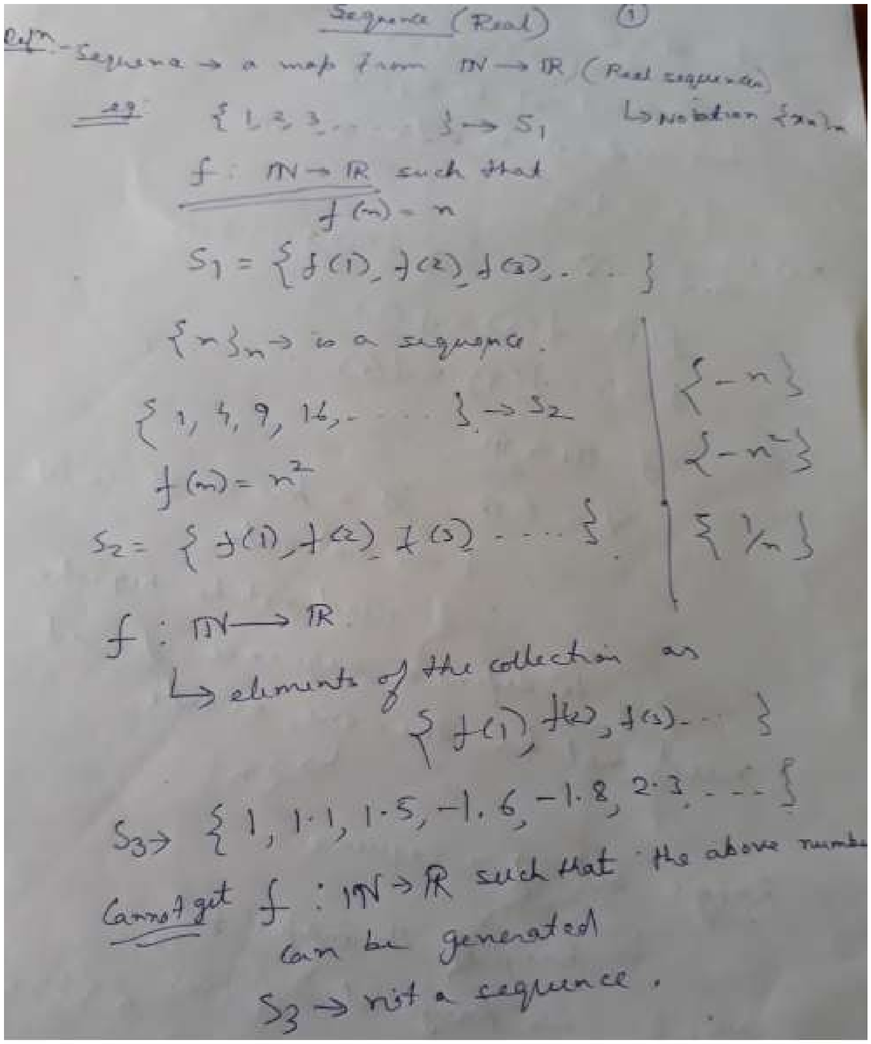
REFERENCE BOOK: **REAL ANALYSIS BY S.K.MAPA**



Sequence (Real) (1)

Defn. - Sequence → a map from $\mathbb{N} \to \mathbb{R}$ (Real sequence)

↳ Notation $\{x_n\}_n$

e.g. $\{1, 2, 3, \ldots\} \to S_1$

$f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = n$

$S_1 = \{f(1), f(2), f(3), \ldots\}$

$\{n\}_n \to$ is a sequence.

$\{-n\}$

$\{-n^2\}$

$\{1/n\}$

$\{1, 4, 9, 16, \ldots\} \to S_2$

$f(n) = n^2$

$S_2 = \{f(1), f(2), f(3) \ldots\}$.

$f : \mathbb{N} \longrightarrow \mathbb{R}$

↳ elements of the collection as $\{f(1), f(2), f(3) \ldots\}$

$S_3 \to \{1, 1.1, 1.5, -1.6, -1.8, 2.3, \ldots\}$

Cannot get $f : \mathbb{N} \to \mathbb{R}$ such that the above numb[ers] can be generated

$S_3 \to$ not a sequence.

Figure 1

$f : \mathbb{N} \to \mathbb{R}$ (sequence)

$f \to$ need not be bijective

↳ need not be onto, one-one

$\{1, -1, 1, -1, \ldots\}$

Not 1-1 $\to \{(-1)^n\} = \{f(1), f(2), f(3) \ldots\}$

$f(1) = f(3) = f(5) \ldots = -1$

$f(2) = f(4) = f(6) \ldots = 1$

$f : \mathbb{N} \to \mathbb{R}$

$\forall x \in \mathbb{R}$, $\nexists$ a pre image in $\mathbb{N}$.

$\therefore f \to$ not onto.

$\therefore f$ need not be bijective.

$\therefore$ So to define a sequence, f need not be one-one, or onto

↳ $\{n\}$.

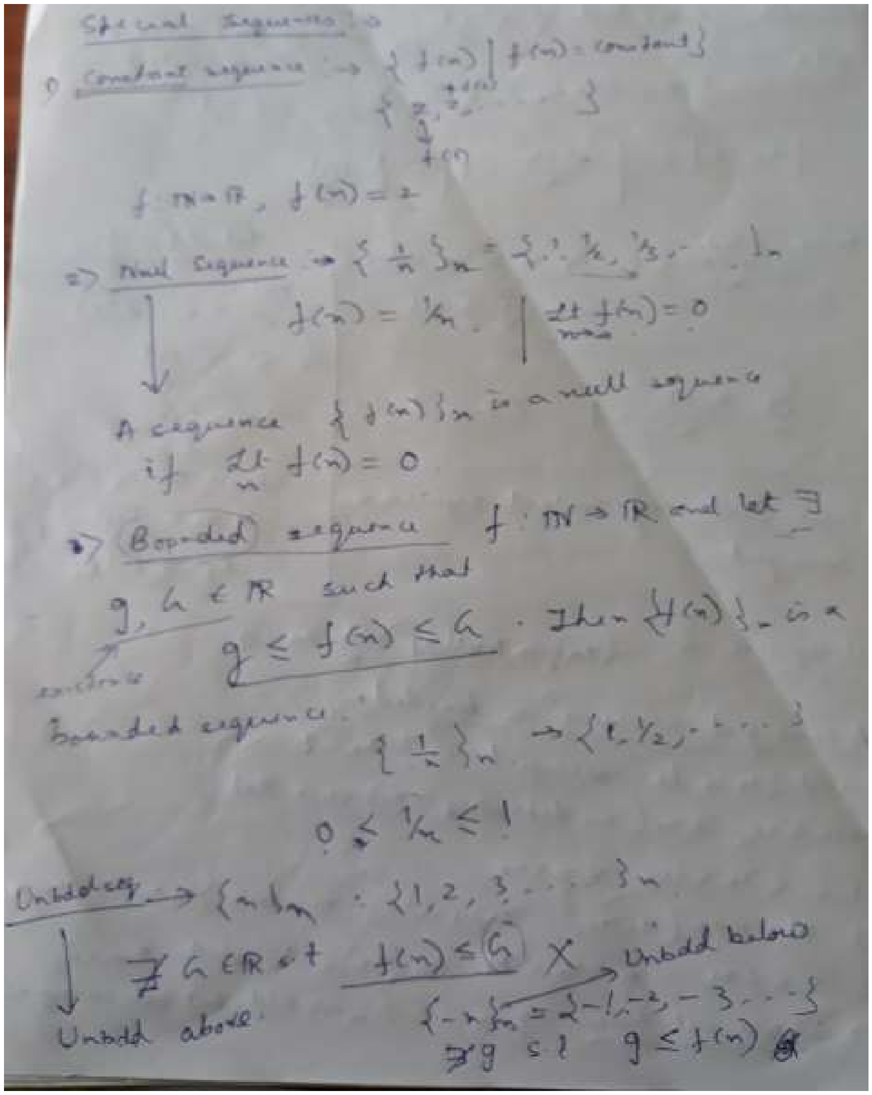
Range set of a sequence:

↳ Range of the map $\{f(n)\}_n$

$\{n\}_n \to$ Range set $= \{f(n)\}_n = \{1, 2, 3, \ldots\}$

$\{-1, 1, -1, 1 \ldots\} \leftarrow \{(-1)^n\}_n \to$ " $= \{-1, 1\}$

Figure 2



Special Sequences :-

1) Constant sequence → $\{f(n) \mid f(n) = \text{constant}\}$

$\{2, 2, \ldots\}$

$f : \mathbb{N} \to \mathbb{R},\ f(n) = 2$

2) Null sequence → $\{\frac{1}{n}\}_n = \{1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}\}$

$f(n) = \frac{1}{n} \quad \Big| \quad \lim_{n\to\infty} f(n) = 0$

A sequence $\{f(n)\}_n$ is a null sequence if $\lim_n f(n) = 0$

3) Bounded sequence $f : \mathbb{N} \to \mathbb{R}$ and let $\exists$ $g, G \in \mathbb{R}$ such that $g \leq f(n) \leq G$. Then $\{f(n)\}_n$ is a Bounded sequence.

constants

$\{\frac{1}{n}\}_n \to \{1, \frac{1}{2}, \ldots\}$

$0 \leq \frac{1}{n} \leq 1$

Unbdd seq → $\{n\}_n : \{1, 2, 3, \ldots\}_n$

$\nexists\ G \in \mathbb{R}$ s.t. $f(n) \leq G$ ✗ → Unbdd below

Unbdd above.

$\{-n\}_n = \{-1, -2, -3, \ldots\}$

$\nexists\ g$ s.t. $g \leq f(n)$

Figure 3

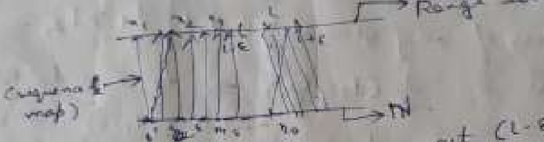
Limit of a sequence

Let $\{x_n\}_n$ be a sequence. $l \in \mathbb{R}$ is said to be limit of the sequence if for any $\epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ (n_0 depending on ϵ) such that

$$|x_n - l| < \epsilon \quad \forall\, n \geq n_0 .$$

note :- $n_0 \to$ depends on ϵ and $\underline{\epsilon \in (0,1) \to \text{arbitrary}}$ so any such number $\in (0,1)$ we can choice.

Idea →



given l, and chosing any $\epsilon > 0 \to$ we get $(l-\epsilon, l+\epsilon)$.
Now if $\exists\, n_0 \in \mathbb{N}$ such that once n_0 is crossed every x_n (with $n \geq n_0$) enters inside $(l-\epsilon, l+\epsilon)$

i.e. come very close to l.
→ arbitrarily close (ϵ).

eg :- $\{\frac{1}{n}\}_n \longrightarrow$ Limit point 0.

Proof :- Here $x_n = \frac{1}{n}$, $l = 0$.

$$|x_n - l| = |\tfrac{1}{n} - 0| = \tfrac{1}{n}$$

Now $\frac{1}{n} < \epsilon \quad \forall\, n \geq n_0$ where $n_0 = [\frac{1}{\epsilon}] + 1$

Calculating n_0

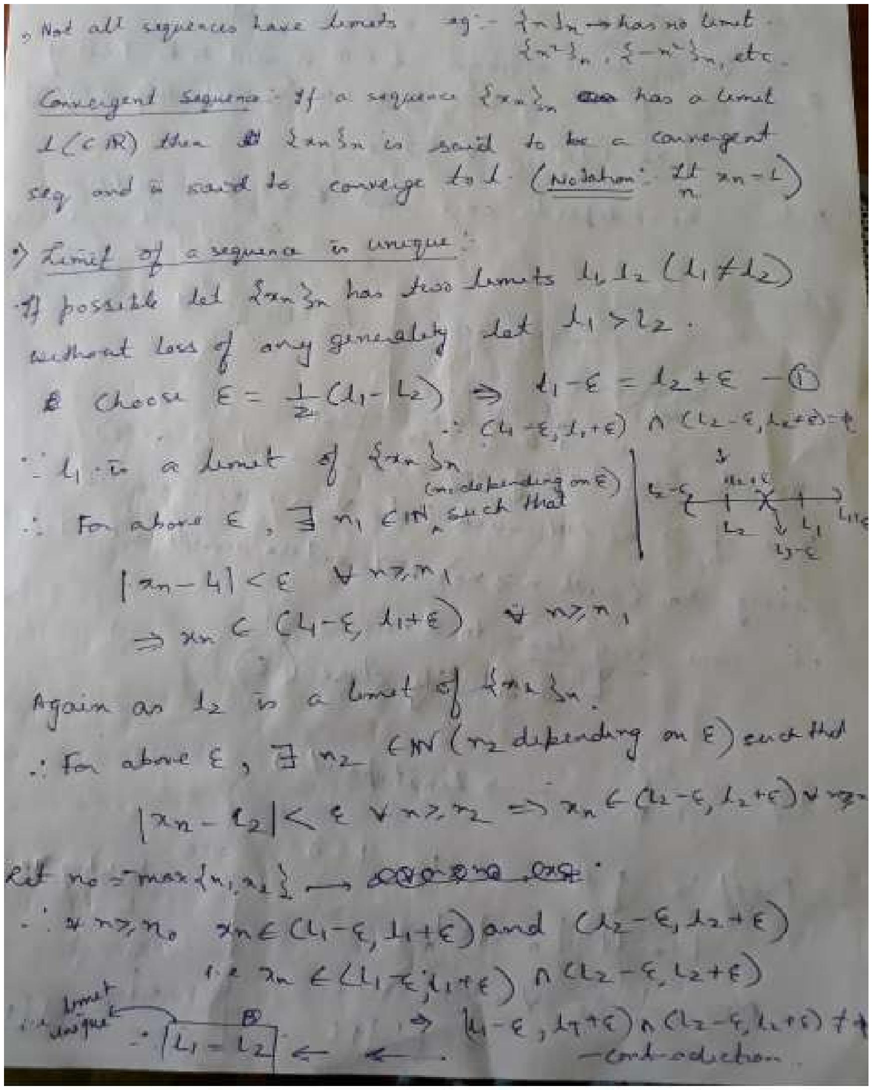
$\frac{1}{n} < \epsilon$

$\Rightarrow n > \frac{1}{\epsilon}$

↑ integer

Choose $n_0 = [\frac{1}{\epsilon}] + 1$

Figure 4



• Not all sequences have limits eg:- $\{n\}_n \to$ has no limit. $\{n^2\}_n$, $\{-n^2\}_n$, etc.

Convergent Sequence: If a sequence $\{x_n\}_n$ has a limit $l (\in \mathbb{R})$ then $\{x_n\}_n$ is said to be a convergent seq and is said to converge to l. (Notation: $\lim_n x_n = l$)

• Limit of a sequence is unique:

If possible let $\{x_n\}_n$ has two limits l_1, l_2 $(l_1 \neq l_2)$

without loss of any generality let $l_1 > l_2$.

Choose $\epsilon = \frac{1}{2}(l_1 - l_2) \Rightarrow l_1 - \epsilon = l_2 + \epsilon$ — (1)

$\therefore (l_1 - \epsilon, l_1 + \epsilon) \cap (l_2 - \epsilon, l_2 + \epsilon) = \phi$

$\because l_1$ is a limit of $\{x_n\}_n$

$\therefore$ For above ϵ, $\exists\, n_1 \in \mathbb{N}$ (n_1 depending on ϵ) such that

$|x_n - l_1| < \epsilon \quad \forall n \geq n_1$

$\Rightarrow x_n \in (l_1 - \epsilon, l_1 + \epsilon) \quad \forall n \geq n_1$

Again as l_2 is a limit of $\{x_n\}_n$.

$\therefore$ For above ϵ, $\exists\, n_2 \in \mathbb{N}$ (n_2 depending on ϵ) such that

$|x_n - l_2| < \epsilon \;\forall n \geq n_2 \Rightarrow x_n \in (l_2 - \epsilon, l_2 + \epsilon) \;\forall n \geq n_2$

Let $n_0 = \max\{n_1, n_2\}$

$\therefore \forall n \geq n_0 \quad x_n \in (l_1 - \epsilon, l_1 + \epsilon)$ and $(l_2 - \epsilon, l_2 + \epsilon)$

i.e. $x_n \in (l_1 - \epsilon, l_1 + \epsilon) \cap (l_2 - \epsilon, l_2 + \epsilon)$

$\Rightarrow (l_1 - \epsilon, l_1 + \epsilon) \cap (l_2 - \epsilon, l_2 + \epsilon) \neq \phi$ — contradiction.

$\therefore \boxed{l_1 = l_2}$ i.e. limit unique

Figure 5

→ Example Prove that $\left\{\frac{n^2+k}{n^2}\right\}_n$ converges to 1 where k is a constant (real)

Solⁿ Here $\{x_n\}_n = \left\{\frac{n^2+k}{n^2}\right\}_n$, $l = 1$

$$|x_n - l| = \left|\frac{n^2+k}{n^2} - 1\right| = \frac{|k|}{n^2}$$

Now $\frac{|k|}{n^2} < \epsilon \quad \forall\, n \geq n_0$ where $n_0 = \sqrt{\frac{|k|}{\epsilon}} + 1$

$\therefore \lim_n \frac{n^2+k}{n^2} = 1$ (Proved.

Calculation of n_0

$\frac{|k|}{n^2} < \epsilon$ — (1)

$\Rightarrow n^2 > \frac{|k|}{\epsilon}$

$\Rightarrow n > \sqrt{\frac{|k|}{\epsilon}}$

choose $n_0 = \sqrt{\frac{|k|}{\epsilon}} + 1$

$\therefore \forall\, n \geq \sqrt{\frac{|k|}{\epsilon}} + 1$, eqⁿ (1) holds

→ Prove that any constant sequence is convergent

Null sequence → $\{x_n\} \to$ Null if $\lim_n x_n = 0$.

→ A convergent sequence is bounded.

Proof → Let $\{x_n\}_n$ be convergent. $\lim_n x_n = l$ (say)

∴ choose $\epsilon > 0$, $\exists\, n_0$ (n_0 depending on ϵ) such that

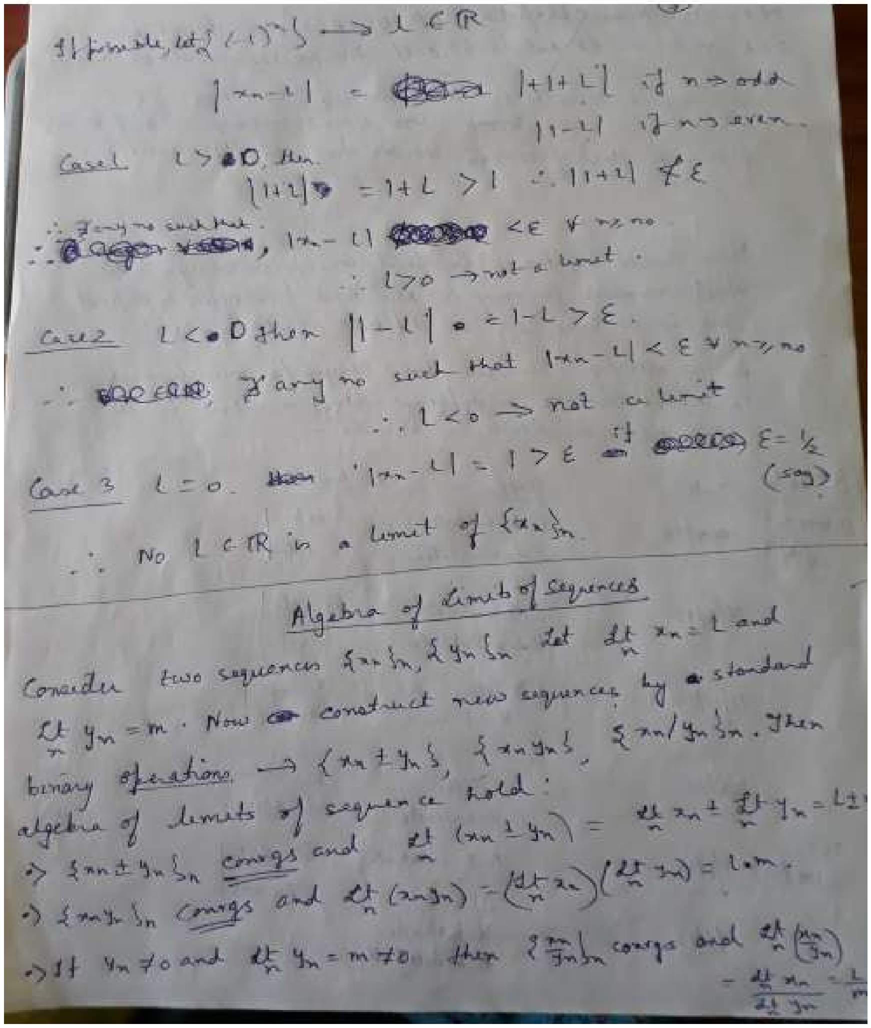
$l - \epsilon < x_n < l + \epsilon \quad \forall\, n \geq n_0$

Let $M = \max\{x_1, x_2, \ldots x_{n_0-1}, l+\epsilon\}$ $m = \min\{x_1, x_2, \ldots x_{n_0-1}, l-\epsilon\}$

$\therefore \forall\, n \in \mathbb{N} \quad m \leq x_n \leq M \Rightarrow \{x_n\}_n$ bounded. Proved.

Note → Converse not true. eg: → $\{(-1)^n\}_n \to$ Bdd but not convergent

Figure 6



If possible, let $\{(-1)^n\} \to l \in \mathbb{R}$

$|x_n - l| = |+1+l|$ if $n \to$ odd

$|1-l|$ if $n \to$ even.

Case 1 $l > 0$, then

$|1+l| = 1+l > 1 \therefore |1+l| \nless \varepsilon$

$\therefore \nexists$ any n_0 such that, $|x_n - l| < \varepsilon \; \forall n \geq n_0$

$\therefore l > 0 \to$ not a limit.

Case 2 $l < 0$ then $|1-l| = 1-l > \varepsilon$.

$\therefore \nexists$ any n_0 such that $|x_n - l| < \varepsilon \; \forall n \geq n_0$

$\therefore l < 0 \Rightarrow$ not a limit

Case 3 $l = 0$. $|x_n - l| = 1 > \varepsilon$ if $\varepsilon = \frac{1}{2}$ (say)

$\therefore$ No $l \in \mathbb{R}$ is a limit of $\{x_n\}_n$

Algebra of Limits of Sequences

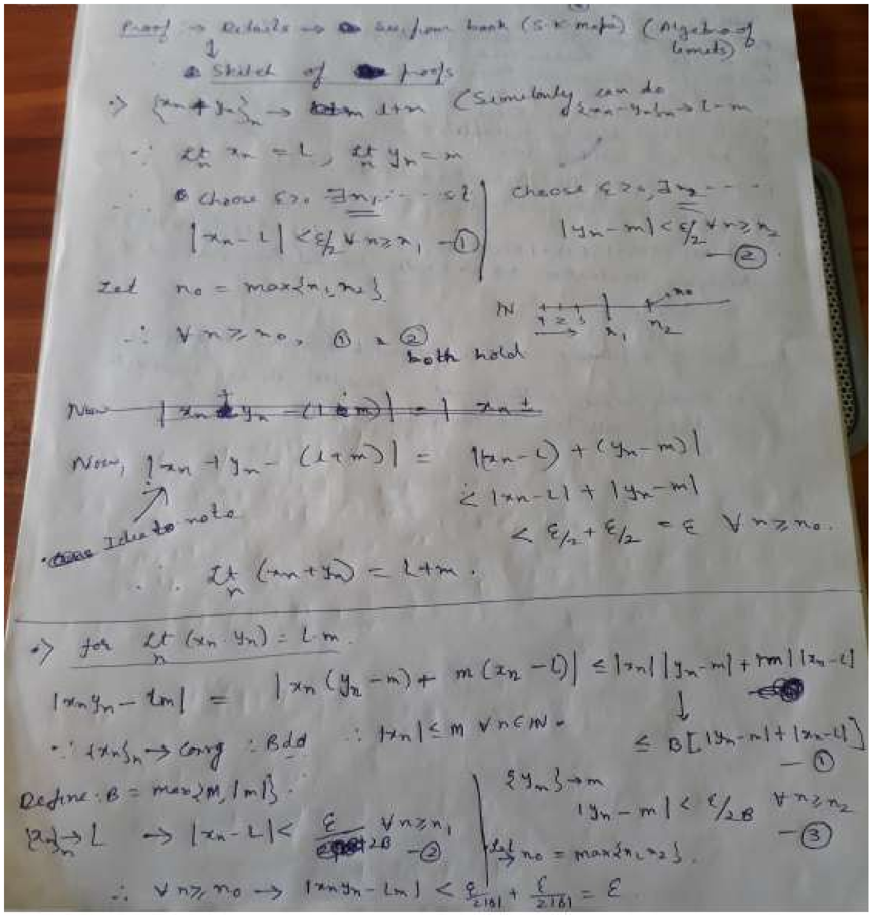
Consider two sequences $\{x_n\}_n, \{y_n\}_n$. Let $\lim_n x_n = l$ and $\lim_n y_n = m$. Now construct new sequences by standard binary operations $\to \{x_n \pm y_n\}, \{x_n y_n\}, \{x_n / y_n\}_n$. Then algebra of limits of sequence hold:

$\to \{x_n \pm y_n\}_n$ convgs and $\lim_n (x_n \pm y_n) = \lim_n x_n \pm \lim_n y_n = l \pm m$

$\to \{x_n y_n\}_n$ convgs and $\lim_n (x_n y_n) = \left(\lim_n x_n\right)\left(\lim_n y_n\right) = l \cdot m$.

$\to$ If $y_n \neq 0$ and $\lim_n y_n = m \neq 0$ then $\left\{\frac{x_n}{y_n}\right\}_n$ convgs and $\lim_n \left(\frac{x_n}{y_n}\right) = \frac{\lim_n x_n}{\lim_n y_n} = \frac{l}{m}$

Figure 7



Proof → Details → See from book (S.K. Mapa) (Algebra of limits)

Sketch of proofs

→ $\{x_n + y_n\}_n \to l+m$ (Similarly can do $\{x_n - y_n\}_n \to l-m$)

∵ $\lim_n x_n = l$, $\lim_n y_n = m$

∴ Choose $\varepsilon > 0$, $\exists n_1 \ldots$ s.t. $|x_n - l| < \varepsilon/2 \; \forall n \geq n_1$ —(1)

Choose $\varepsilon > 0$, $\exists n_2 \ldots$ $|y_n - m| < \varepsilon/2 \; \forall n \geq n_2$ —(2)

Let $n_0 = \max\{n_1, n_2\}$

∴ $\forall n \geq n_0$, (1) & (2) both hold

Now, $|x_n + y_n - (l+m)| = |(x_n - l) + (y_n - m)| \leq |x_n - l| + |y_n - m| < \varepsilon/2 + \varepsilon/2 = \varepsilon \; \forall n \geq n_0$.

Idea to note

∴ $\lim_n (x_n + y_n) = l+m$.

→ for $\lim_n (x_n \cdot y_n) = l \cdot m$.

$|x_n y_n - lm| = |x_n(y_n - m) + m(x_n - l)| \leq |x_n||y_n - m| + |m||x_n - l| \leq B[|y_n - m| + |x_n - l|]$ —(1)

∵ $\{x_n\}_n \to$ conv ∴ Bdd ∴ $|x_n| \leq M \; \forall n \in \mathbb{N}$.

Define $B = \max\{M, |m|\}$.

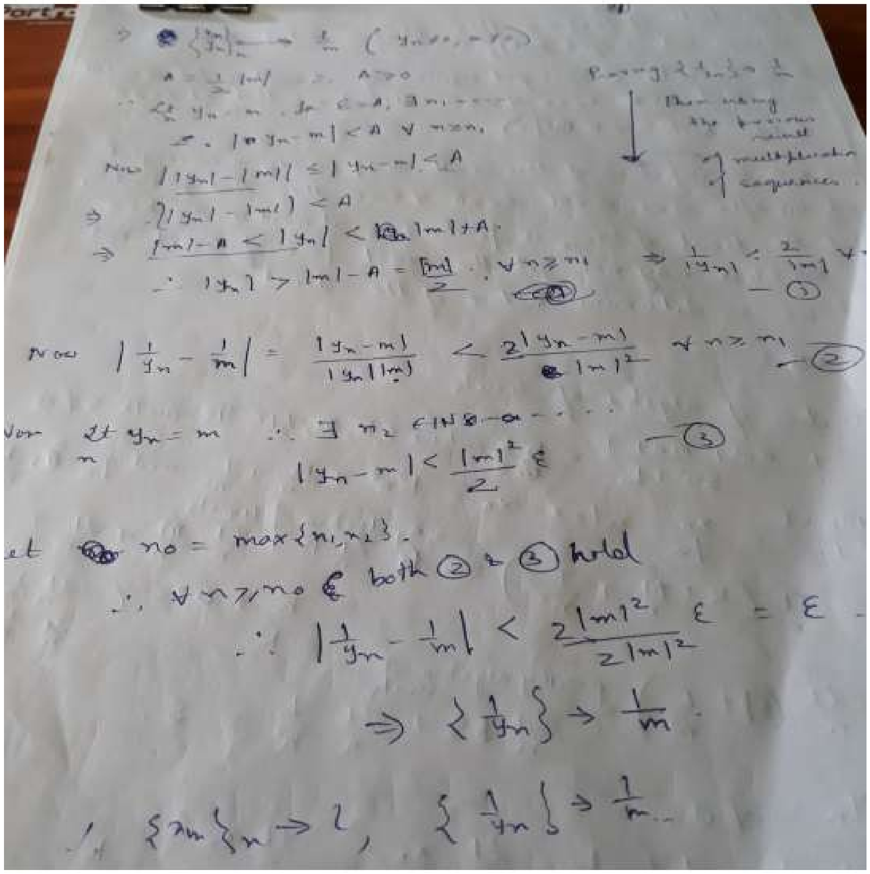
$\{x_n\}_n \to l \to |x_n - l| < \frac{\varepsilon}{2B} \; \forall n \geq n_1$ —(2)

$\{y_n\} \to m$, $|y_n - m| < \varepsilon/2B \; \forall n \geq n_2$ —(3)

Let $n_0 = \max\{n_1, n_2\}$.

∴ $\forall n \geq n_0 \to |x_n y_n - lm| < \frac{\varepsilon}{2|B|} + \frac{\varepsilon}{2|B|} = \varepsilon$

Figure 8



$A = \frac{1}{2}|m|$ ∴ $A > 0$

$|y_n| > |m| - A = \frac{|m|}{2}$ ∀ $n \geq n_1$

Now $\left|\frac{1}{y_n} - \frac{1}{m}\right| = \frac{|y_n - m|}{|y_n||m|} < \frac{2|y_n - m|}{|m|^2}$ ∀ $n \geq n_1$ — (2)

$|y_n - m| < \frac{|m|^2}{2}\epsilon$ — (3)

let $n_0 = \max\{n_1, n_2\}$.

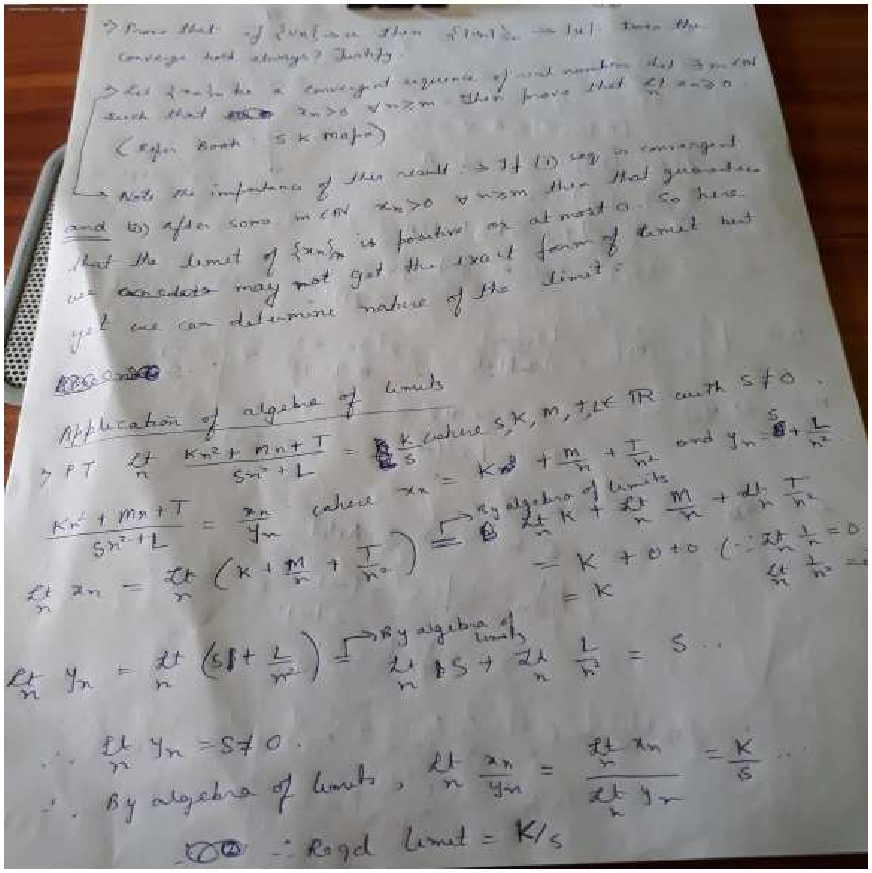
∴ ∀ $n \geq n_0$ both (2) & (3) hold

∴ $\left|\frac{1}{y_n} - \frac{1}{m}\right| < \frac{2|m|^2}{2|m|^2}\epsilon = \epsilon$

$\Rightarrow \left\{\frac{1}{y_n}\right\} \to \frac{1}{m}$

∴ $\{x_n\}_n \to l$, $\left\{\frac{1}{y_n}\right\} \to \frac{1}{m}$

Figure 9



→ Prove that if $\{x_n\}_n \to l$ then $\{|x_n|\}_n \to |l|$. Does the converse hold always? Justify.

→ Let $\{x_n\}_n$ be a convergent sequence of real numbers that $\exists\, m \in \mathbb{N}$ such that $x_n > 0\ \forall n \geq m$. Then prove that $\lim_n x_n \geq 0$.

(Refer Book : S.K Mapa)

→ Note the importance of this result → If (i) seq is convergent and (ii) after some $m \in \mathbb{N}$ $x_n > 0\ \forall n \geq m$ then that guarantees that the limit of $\{x_n\}_n$ is positive or at most 0. So here we may not get the exact form of limit but yet we can determine nature of the limit.

Application of algebra of limits

→ PT $\lim_n \frac{Kn^2 + Mn + T}{Sn^2 + L} = \frac{K}{S}$ where $S, K, M, T, L \in \mathbb{R}$ with $S \neq 0$.

$\frac{Kn^2 + Mn + T}{Sn^2 + L} = \frac{x_n}{y_n}$ where $x_n = K + \frac{M}{n} + \frac{T}{n^2}$ and $y_n = S + \frac{L}{n^2}$

$\lim_n x_n = \lim_n \left(K + \frac{M}{n} + \frac{T}{n^2}\right) \overset{\text{By algebra of limits}}{=} \lim_n K + \lim_n \frac{M}{n} + \lim_n \frac{T}{n^2}$

$= K + 0 + 0$ ($\because \lim_n \frac{1}{n} = 0$, $\lim_n \frac{1}{n^2} = 0$)

$= K$

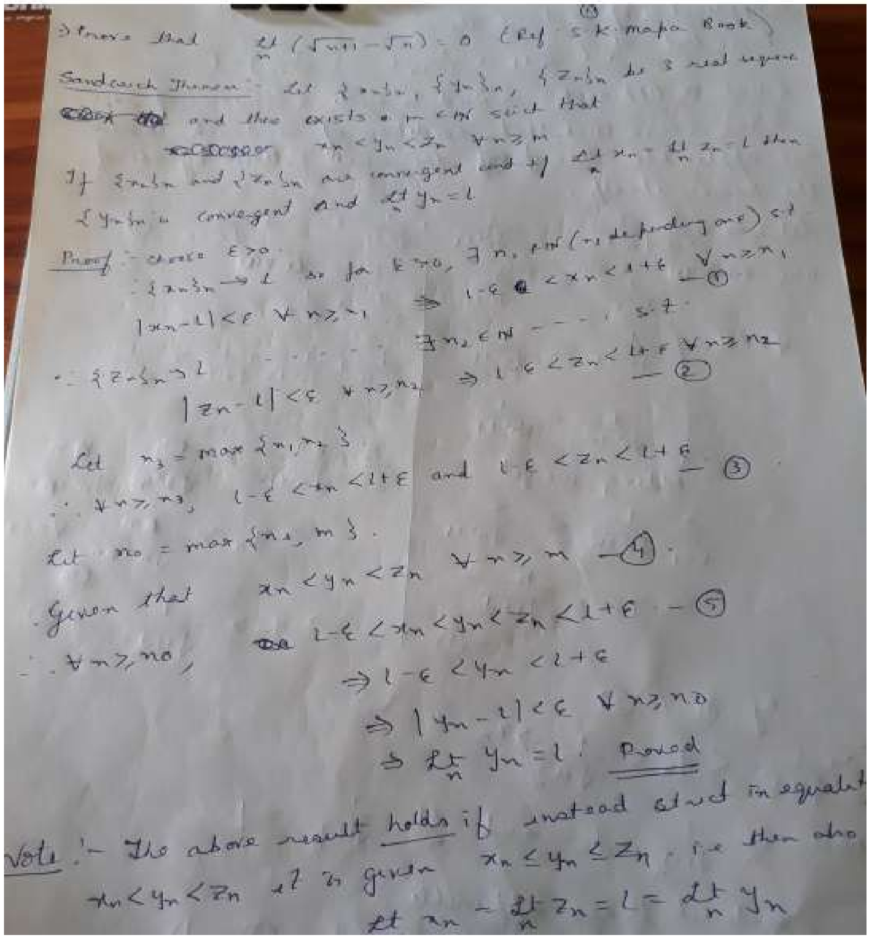
$\lim_n y_n = \lim_n \left(S + \frac{L}{n^2}\right) \overset{\text{By algebra of limits}}{=} \lim_n S + \lim_n \frac{L}{n^2} = S$

$\therefore \lim_n y_n = S \neq 0$.

$\therefore$ By algebra of limits, $\lim_n \frac{x_n}{y_n} = \frac{\lim_n x_n}{\lim_n y_n} = \frac{K}{S}$

$\therefore$ Reqd limit $= K/S$

Figure 10



⇒ Prove that $\underset{n}{Lt}(\sqrt{n+1}-\sqrt{n}) = 0$ (Ref S.K. Mapa Book)

Sandwich Theorem :- Let $\{x_n\}_n$, $\{y_n\}_n$, $\{z_n\}_n$ be 3 real sequence and there exists $m \in \mathbb{N}$ such that

$x_n < y_n < z_n \quad \forall n \geq m$

If $\{x_n\}_n$ and $\{z_n\}_n$ are convergent and if $\underset{n}{Lt}\, x_n = \underset{n}{Lt}\, z_n = l$ then $\{y_n\}_n$ is convergent and $Lt\, y_n = l$

Proof :- choose $\epsilon > 0$.

$\because \{x_n\}_n \to l$ so for $\epsilon > 0$, $\exists n_1 \in \mathbb{N}$ (n_1 depending on ϵ) s.t.

$|x_n - l| < \epsilon \quad \forall n \geq n_1 \Rightarrow l-\epsilon < x_n < l+\epsilon \quad \forall n \geq n_1$ — ①

$\because \{z_n\}_n \to l$ ... $\exists n_2 \in \mathbb{N}$ ... s.t.

$|z_n - l| < \epsilon \quad \forall n \geq n_2 \Rightarrow l-\epsilon < z_n < l+\epsilon \quad \forall n \geq n_2$ — ②

Let $n_3 = \max\{n_1, n_2\}$

$\therefore \forall n \geq n_3$, $l-\epsilon < x_n < l+\epsilon$ and $l-\epsilon < z_n < l+\epsilon$ — ③

Let $n_0 = \max\{n_3, m\}$.

Given that $x_n < y_n < z_n \quad \forall n \geq m$ — ④.

$\therefore \forall n \geq n_0$, $l-\epsilon < x_n < y_n < z_n < l+\epsilon$ — ⑤

$\Rightarrow l-\epsilon < y_n < l+\epsilon$

$\Rightarrow |y_n - l| < \epsilon \quad \forall n \geq n_0$

$\Rightarrow \underset{n}{Lt}\, y_n = l$. Proved

Note :- The above result holds if instead strict inequality $x_n < y_n < z_n$ it is given $x_n \leq y_n \leq z_n$ i.e. then also $Lt\, x_n = \underset{n}{Lt}\, z_n = l = \underset{n}{Lt}\, y_n$

Figure 11

(How to Approach → Need to create 3 sequences $\{x_n\}_n$, $\{y_n\}$, $\{z_n\}_n$ such that condition of Sandwich theorem are applicable then the result follows by use of the theorem itself.

Let $y_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}$ — ①

Now $\frac{1}{\sqrt{n^2+1}} = \frac{1}{\sqrt{n^2+1}}$

$\forall n \in \mathbb{N}$

$\frac{1}{\sqrt{n^2+2}} < \frac{1}{\sqrt{n^2+1}}$

$\frac{1}{\sqrt{n^2+3}} < \frac{1}{\sqrt{n^2+1}}$

$\frac{1}{\sqrt{n^2+n}} < \frac{1}{\sqrt{n^2+1}}$

∴ Adding, we get

$\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} < \frac{n}{\sqrt{n^2+1}}$

$\Rightarrow y_n < \frac{\cancel{n}}{\cancel{n}\sqrt{1+\frac{1}{n^2}}}$ — ①

Again,

$\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} > \frac{2}{\sqrt{n^2+2}}$

$\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} > \frac{3}{\sqrt{n^2+3}}$

$\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} > \frac{n}{\sqrt{n^2+n}}$

$\Rightarrow y_n > \frac{n}{\sqrt{n^2+n}}$ $\forall n \geq 2$

$\Rightarrow y_n > \frac{1}{\sqrt{1+\frac{1}{n}}}$ — ②

∴ By ① & ②

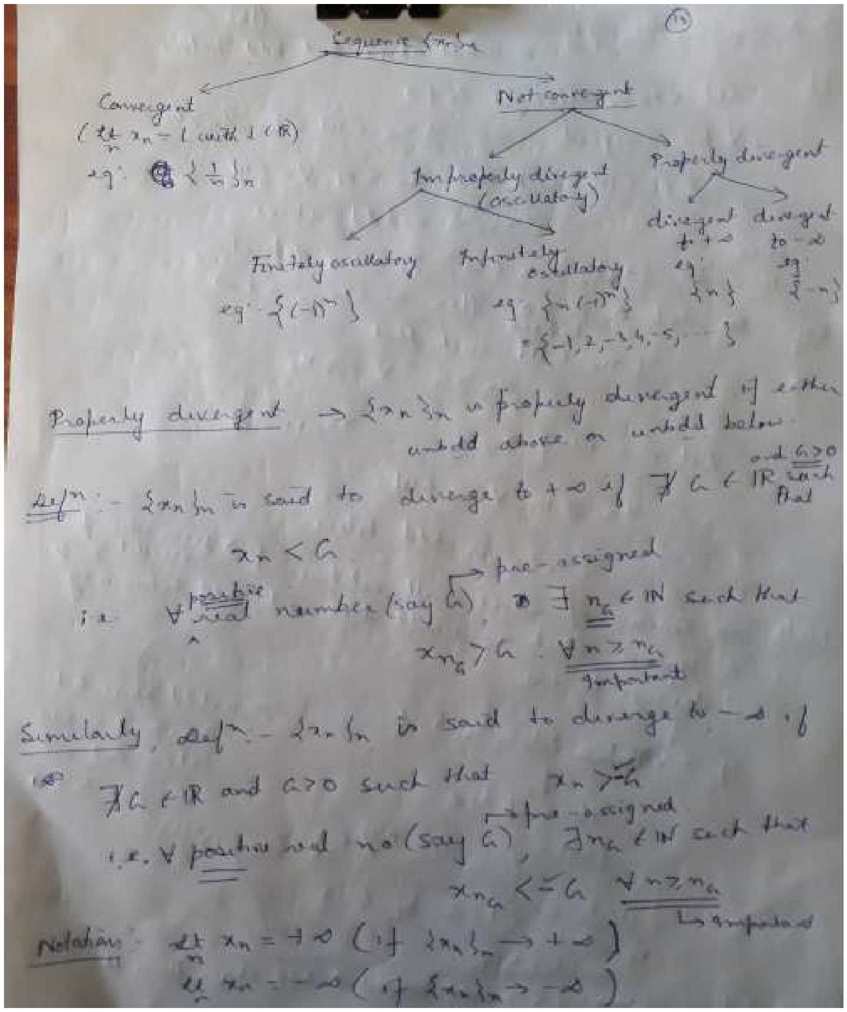
$x_n \to \frac{1}{\sqrt{1+\frac{1}{n}}} < y_n < \frac{1}{\sqrt{1+\frac{1}{n^2}}} \to z_n$

Now $\lim_{n} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$ and $\lim_{n} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$.

∴ Sandwich Theorem applicable.

∴ By Sandwich Theorem $\lim_{n} y_n = 1$ (Proved)

Figure 12



Sequence $\{x_n\}_n$

Convergent ($\lim_n x_n = l$ with $l \in \mathbb{R}$)
eg: $\{\frac{1}{n}\}_n$

Not convergent

Improperly divergent (oscillatory)

Finitely oscillatory
eg: $\{(-1)^n\}$

Infinitely oscillatory
eg: $\{n(-1)^n\}$ $= \{-1, 2, -3, 4, -5, \dots\}$

Properly divergent

divergent to $+\infty$
eg: $\{n\}$

divergent to $-\infty$
eg: $\{-n\}$

Properly divergent → $\{x_n\}_n$ is properly divergent if either unbdd above or unbdd below.

Defn:- $\{x_n\}_n$ is said to diverge to $+\infty$ if $\nexists\, G \in \mathbb{R}$ and $G > 0$ such that

$x_n < G$

i.e. $\forall$ positive real number (say G) → pre-assigned, $\exists\, n_G \in \mathbb{N}$ such that

$x_{n_G} > G \quad \forall n \geq n_G$ important

Similarly, Defn:- $\{x_n\}_n$ is said to diverge to $-\infty$ if

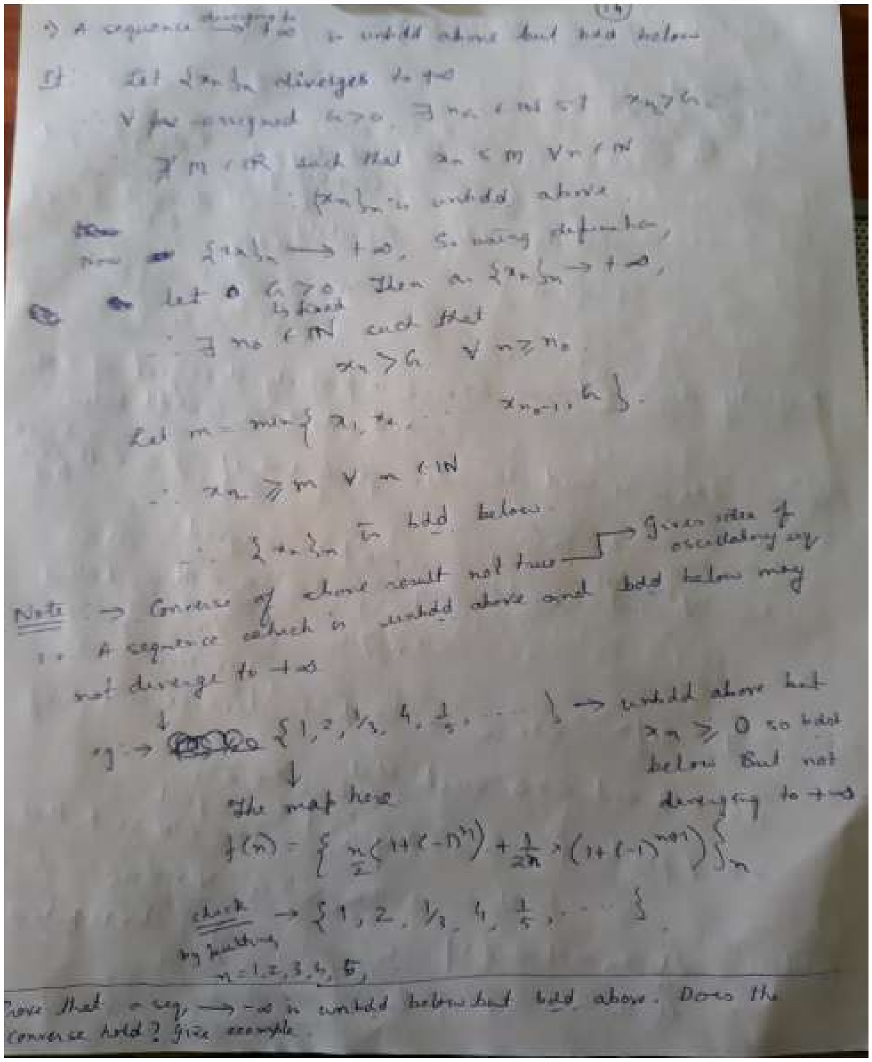
$\nexists\, G \in \mathbb{R}$ and $G > 0$ such that $x_n > -G$

i.e. $\forall$ positive real no (say G) → pre-assigned, $\exists\, n_G \in \mathbb{N}$ such that

$x_{n_G} < -G \quad \forall n \geq n_G$ → important

Notation: $\lim_n x_n = +\infty$ (if $\{x_n\}_n \to +\infty$)
$\lim_n x_n = -\infty$ (if $\{x_n\}_n \to -\infty$)

Figure 13



(14)

→ A sequence $\xrightarrow{\text{diverging to}} +\infty$ is unbdd above but bdd below

St Let $\{x_n\}_n$ diverges to $+\infty$

$\forall$ pre-assigned $G > 0$, $\exists\, n_0 \in \mathbb{N}$ s.t. $x_{n_0} > G$.

$\nexists\, M \in \mathbb{R}$ such that $x_n \le M \;\forall n \in \mathbb{N}$

$\therefore \{x_n\}_n$ is unbdd above.

now $\{x_n\}_n \longrightarrow +\infty$, So using definition,

let $G > 0$ is fixed. Then as $\{x_n\}_n \to +\infty$,

$\therefore \exists\, n_0 \in \mathbb{N}$ such that

$x_n > G \;\; \forall n \ge n_0$.

Let $m = \min\{x_1, x_2, \ldots x_{n_0-1}, G\}$.

$\therefore x_n \geq m \;\; \forall n \in \mathbb{N}$

$\therefore \{x_n\}_n$ is bdd below.

Note → Converse of above result not true → Gives idea of oscillatory seq.

∴ A sequence which is unbdd above and bdd below may not diverge to $+\infty$

eg → $\{1, 2, 1/3, 4, \frac{1}{5}, \ldots\}$ → unbdd above but $x_n \geq 0$ so bdd below But not diverging to $+\infty$

The map here

$$f(n) = \left\{ \frac{n}{2}(1+(-1)^n) + \frac{1}{2n}\times(1+(-1)^{n+1}) \right\}_n$$

check → $\{1, 2, 1/3, 4, \frac{1}{5}, \ldots\}$

by putting $n = 1, 2, 3, 4, 5,$

Prove that a seq. $\longrightarrow -\infty$ is unbdd below but bdd above. Does the converse hold? Give example.

Figure 14