SEMESTER-IV

LECTURE NOTES ON

Simple harmonic motion

1st PART

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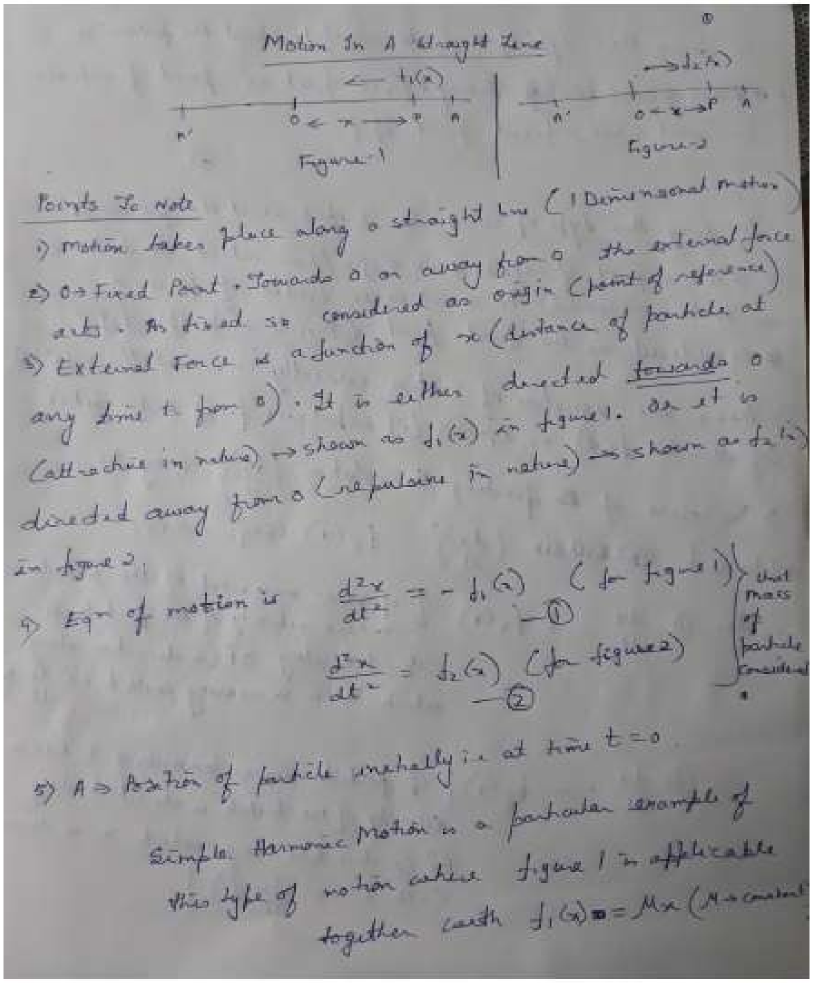
REFERENCE BOOK: **PARTICLE DYNAMICS BY SAHA AND GANGULY**



Motion In A Straight Line

Figure 1 | Figure 2

Points To Note

1) Motion takes place along a straight line (1 Dimensional motion)

2) O → Fixed Point. Towards O or away from O the external force acts. The fixed is considered as origin (point of reference)

3) External Force is a function of x (distance of particle at any time t from O). It is either directed towards O in figure 1. (attractive in nature) → shown as $f_1(x)$ or it is directed away from O (repulsive in nature) → shown as $f_2(x)$ in figure 2.

4) Eqn of motion is $\frac{d^2x}{dt^2} = -f_1(x)$ (for figure 1) — (1)

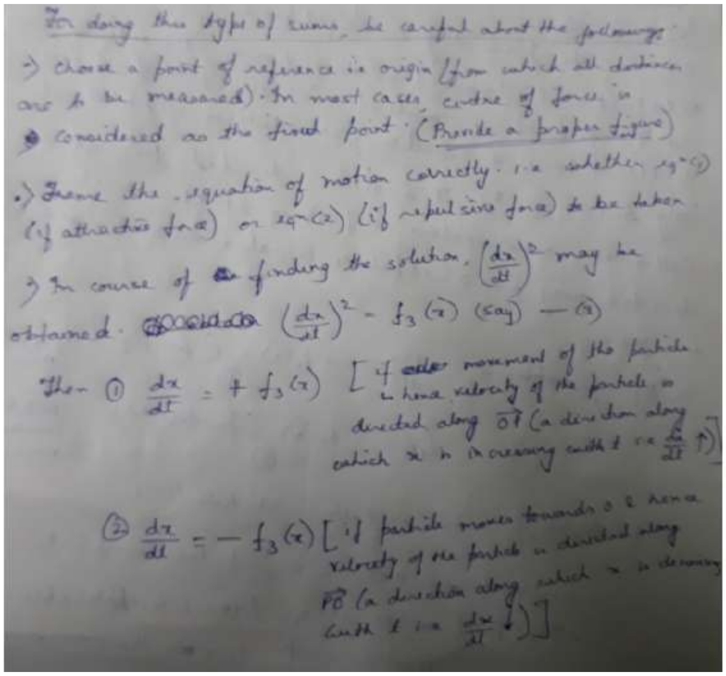
$\frac{d^2x}{dt^2} = f_2(x)$ (for figure 2) — (2)

unit mass of particle considered

5) A → Position of particle initially i.e. at time $t = 0$

Simple Harmonic Motion is a particular example of this type of motion where figure 1 is applicable together with $f_1(x) = Mx$ (M = constant

Figure 1



For doing this type of sums, be careful about the following:

→ Choose a point of reference i.e origin (from which all distances are to be measured). In most cases, centre of force is considered as the fixed point. (Provide a proper figure)

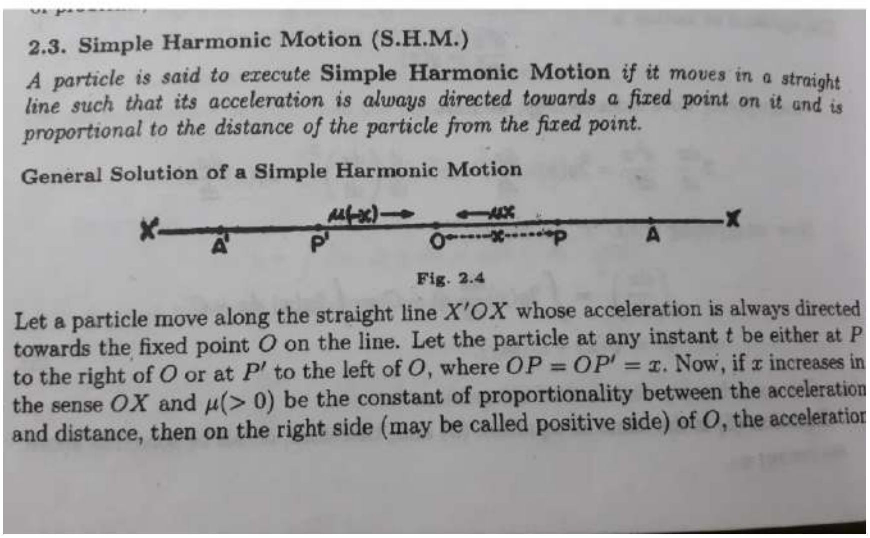
→ Frame the equation of motion correctly. i.e whether eqn (1) (if attractive force) or eqn (2) (if repulsive force) to be taken

→ In course of finding the solution, $\left(\frac{dx}{dt}\right)^2$ may be obtained. $\left(\frac{dx}{dt}\right)^2 = f_3(x)$ (say) — (3)

Then (1) $\frac{dx}{dt} = + f_3(x)$ [if movement of the particle & hence velocity of the particle is directed along $\vec{OP}$ (a direction along which x is increasing with t i.e $\frac{dx}{dt}\uparrow$)]

(2) $\frac{dx}{dt} = - f_3(x)$ [if particle moves towards O & hence velocity of the particle is directed along $\vec{PO}$ (a direction along which x is decreasing with t i.e $\frac{dx}{dt}\downarrow$)]

Figure 2



2.3. Simple Harmonic Motion (S.H.M.)

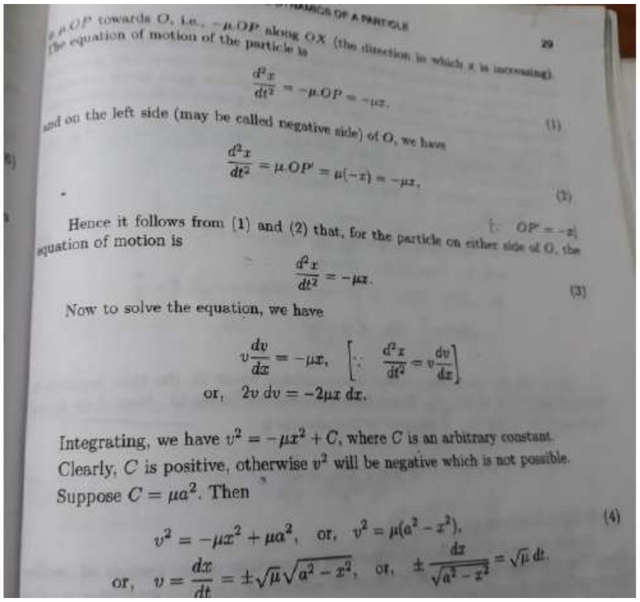
A particle is said to execute **Simple Harmonic Motion** *if it moves in a straight line such that its acceleration is always directed towards a fixed point on it and is proportional to the distance of the particle from the fixed point.*

General Solution of a Simple Harmonic Motion

Fig. 2.4

Let a particle move along the straight line $X'OX$ whose acceleration is always directed towards the fixed point O on the line. Let the particle at any instant t be either at P to the right of O or at P' to the left of O, where $OP = OP' = x$. Now, if x increases in the sense OX and $\mu(> 0)$ be the constant of proportionality between the acceleration and distance, then on the right side (may be called positive side) of O, the acceleration

Figure 3



DYNAMICS OF A PARTICLE 29

$\mu.OP$ towards O, i.e., $-\mu.OP$ along OX (the direction in which x is increasing). The equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\mu.OP = -\mu x. \qquad (1)$$

and on the left side (may be called negative side) of O, we have

$$\frac{d^2x}{dt^2} = \mu.OP' = \mu(-x) = -\mu x, \qquad (2)$$

$[\because OP' = -x]$

Hence it follows from (1) and (2) that, for the particle on either side of O, the equation of motion is

$$\frac{d^2x}{dt^2} = -\mu x. \qquad (3)$$

Now to solve the equation, we have

$$v\frac{dv}{dx} = -\mu x, \quad \left[\because \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right]$$

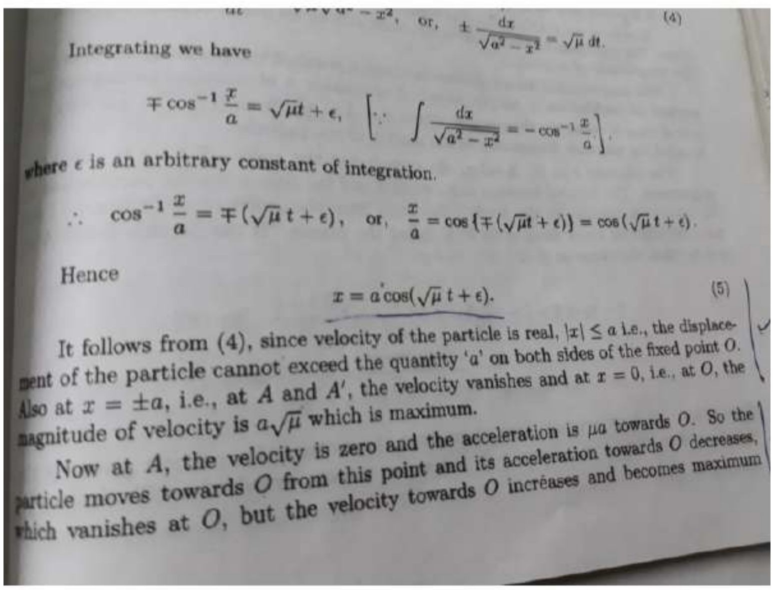
$$\text{or,} \quad 2v\,dv = -2\mu x\,dx.$$

Integrating, we have $v^2 = -\mu x^2 + C$, where C is an arbitrary constant. Clearly, C is positive, otherwise v^2 will be negative which is not possible. Suppose $C = \mu a^2$. Then

$$v^2 = -\mu x^2 + \mu a^2, \quad \text{or,} \quad v^2 = \mu(a^2 - x^2). \qquad (4)$$

$$\text{or,} \quad v = \frac{dx}{dt} = \pm\sqrt{\mu}\sqrt{a^2 - x^2}, \quad \text{or,} \quad \pm\frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\mu}\,dt.$$

Figure 4



$$\text{or,} \quad \pm\frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\mu}\,dt. \qquad (4)$$

Integrating we have

$$\mp\cos^{-1}\frac{x}{a} = \sqrt{\mu}t + \epsilon, \quad \left[\because \int\frac{dx}{\sqrt{a^2 - x^2}} = -\cos^{-1}\frac{x}{a}\right].$$

where ϵ is an arbitrary constant of integration.

$$\therefore \quad \cos^{-1}\frac{x}{a} = \mp(\sqrt{\mu}\,t + \epsilon), \quad \text{or,} \quad \frac{x}{a} = \cos\{\mp(\sqrt{\mu}t + \epsilon)\} = \cos(\sqrt{\mu}\,t + \epsilon).$$

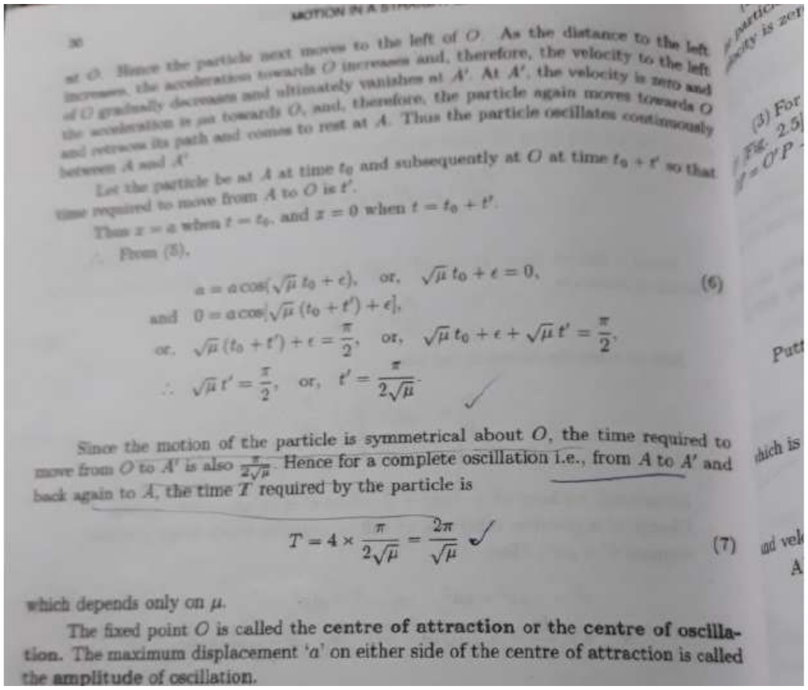
Hence

$$x = a\cos(\sqrt{\mu}\,t + \epsilon). \qquad (5)$$

It follows from (4), since velocity of the particle is real, $|x| \le a$ i.e., the displacement of the particle cannot exceed the quantity 'a' on both sides of the fixed point O. Also at $x = \pm a$, i.e., at A and A', the velocity vanishes and at $x = 0$, i.e., at O, the magnitude of velocity is $a\sqrt{\mu}$ which is maximum.

Now at A, the velocity is zero and the acceleration is μa towards O. So the particle moves towards O from this point and its acceleration towards O decreases, which vanishes at O, but the velocity towards O increases and becomes maximum

Figure 5



30 MOTION IN A ST...

at O. Hence the particle next moves to the left of O. As the distance to the left increases, the acceleration towards O increases and, therefore, the velocity to the left of O gradually decreases and ultimately vanishes at A'. At A', the velocity is zero and the acceleration is just towards O, and, therefore, the particle again moves towards O and retraces its path and comes to rest at A. Thus the particle oscillates continuously between A and A'.

Let the particle be at A at time t_0 and subsequently at O at time $t_0 + t'$ so that time required to move from A to O is t'.

Thus $x = a$ when $t = t_0$, and $x = 0$ when $t = t_0 + t'$.

From (5),

$$a = a\cos(\sqrt{\mu}\, t_0 + \epsilon), \quad \text{or,} \quad \sqrt{\mu}\, t_0 + \epsilon = 0, \tag{6}$$

$$\text{and} \quad 0 = a\cos[\sqrt{\mu}\,(t_0 + t') + \epsilon],$$

$$\text{or,} \quad \sqrt{\mu}\,(t_0 + t') + \epsilon = \frac{\pi}{2}, \quad \text{or,} \quad \sqrt{\mu}\, t_0 + \epsilon + \sqrt{\mu}\, t' = \frac{\pi}{2},$$

$$\therefore \quad \sqrt{\mu}\, t' = \frac{\pi}{2}, \quad \text{or,} \quad t' = \frac{\pi}{2\sqrt{\mu}}.$$

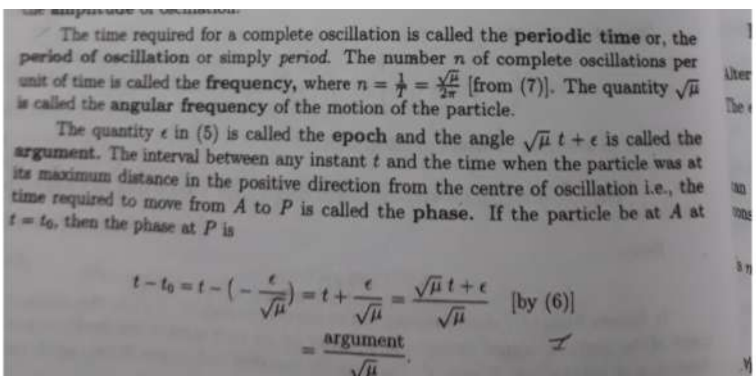
Since the motion of the particle is symmetrical about O, the time required to move from O to A' is also $\frac{\pi}{2\sqrt{\mu}}$. Hence for a complete oscillation i.e., from A to A' and back again to A, the time T required by the particle is

$$T = 4 \times \frac{\pi}{2\sqrt{\mu}} = \frac{2\pi}{\sqrt{\mu}} \tag{7}$$

which depends only on μ.

The fixed point O is called the **centre of attraction** or the **centre of oscillation**. The maximum displacement 'a' on either side of the centre of attraction is called the **amplitude of oscillation**.

Figure 6



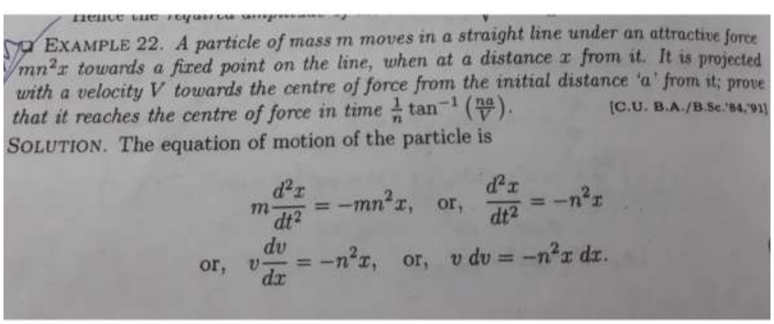
The time required for a complete oscillation is called the **periodic time** or, the **period of oscillation** or simply *period*. The number n of complete oscillations per unit of time is called the **frequency**, where $n = \frac{1}{T} = \frac{\sqrt{\mu}}{2\pi}$ [from (7)]. The quantity $\sqrt{\mu}$ is called the **angular frequency** of the motion of the particle.

The quantity ϵ in (5) is called the **epoch** and the angle $\sqrt{\mu}\, t + \epsilon$ is called the **argument**. The interval between any instant t and the time when the particle was at its maximum distance in the positive direction from the centre of oscillation i.e., the time required to move from A to P is called the **phase**. If the particle be at A at $t = t_0$, then the phase at P is

$$t - t_0 = t - \left(-\frac{\epsilon}{\sqrt{\mu}}\right) = t + \frac{\epsilon}{\sqrt{\mu}} = \frac{\sqrt{\mu}\, t + \epsilon}{\sqrt{\mu}} \quad \text{[by (6)]}$$

$$= \frac{\text{argument}}{\sqrt{\mu}}.$$

Figure 7



EXAMPLE 22. *A particle of mass m moves in a straight line under an attractive force mn^2x towards a fixed point on the line, when at a distance x from it. It is projected with a velocity V towards the centre of force from the initial distance 'a' from it; prove that it reaches the centre of force in time $\frac{1}{n}\tan^{-1}\left(\frac{na}{V}\right)$.* [C.U. B.A./B.Sc.'84,'91]

SOLUTION. The equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = -mn^2x, \quad \text{or,} \quad \frac{d^2x}{dt^2} = -n^2x$$

$$\text{or,} \quad v\frac{dv}{dx} = -n^2x, \quad \text{or,} \quad v\,dv = -n^2x\,dx.$$

Figure 8

Integrating,

$$\int v\,dv = -n^2 \int x\,dx, \quad \text{or,} \quad \frac{v^2}{2} = -n^2\,\frac{x^2}{2} + C, \qquad (2)$$

where C is an arbitrary constant.

Initially, at $t = 0$, $x = a$ and $v = -V$;

$$\therefore \quad \frac{V^2}{2} = -\frac{n^2a^2}{2} + C, \quad \text{or,} \quad C = \frac{1}{2}(V^2 + n^2a^2)$$

$\therefore$ From (2),

$$\frac{v^2}{2} = -\frac{n^2x^2}{2} + \frac{1}{2}(V^2 + n^2a^2)$$

or, $v^2 = -n^2x^2 + V^2 + n^2a^2$, or, $v = -\sqrt{(V^2 + n^2a^2) - n^2x^2}$

or, $\dfrac{dx}{dt} = -\sqrt{(V^2 + n^2a^2) - n^2x^2}$

[negative sign is taken, since the velocity is towards the fixed point O and $\frac{dx}{dt}$ is towards x increasing]

or, $dt = -\dfrac{dx}{n\sqrt{\frac{V^2}{n^2} + a^2 - x^2}} = -\dfrac{dx}{n\sqrt{b^2 - x^2}}$, where $b^2 = \dfrac{V^2}{n^2} + a^2$

Integrating

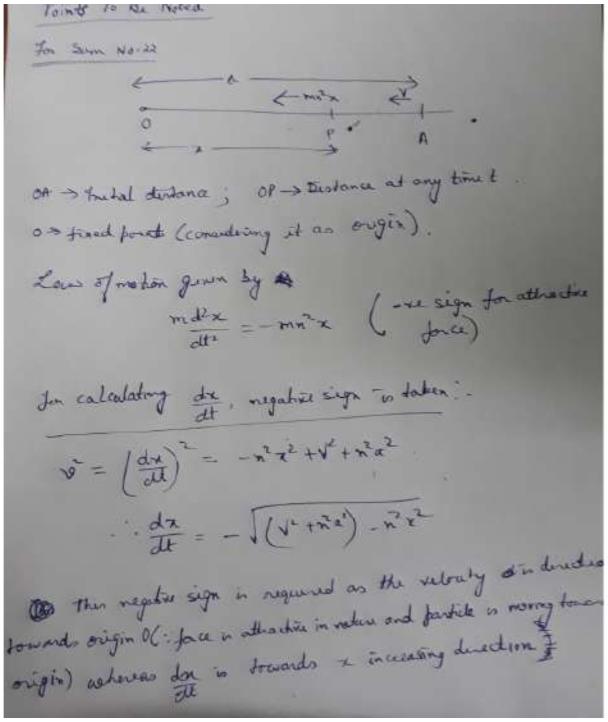
$$\int_0^T dt = -\frac{1}{n}\int_a^0 \frac{dx}{\sqrt{b^2 - x^2}},$$ where T is the required time from $x = a$ to $x = 0$.

or, $T = -\dfrac{1}{n}\left[\sin^{-1}\dfrac{x}{b}\right]_a^0 = -\dfrac{1}{n}\left[0 - \sin^{-1}\dfrac{a}{b}\right]$, where $b^2 = \dfrac{V^2}{n^2} + a^2$

$= \dfrac{1}{n}\sin^{-1}\dfrac{a}{b} \quad \left[\text{If } \sin^{-1}\dfrac{a}{b} = \alpha, \text{ then } \sin\alpha = \dfrac{a}{b}; \quad \therefore \quad \tan\alpha = \dfrac{a}{\sqrt{b^2 - a^2}}\right]$

$= \dfrac{1}{n}\tan^{-1}\dfrac{a}{\sqrt{b^2 - a^2}} = \dfrac{1}{n}\tan^{-1}\left(\dfrac{na}{V}\right) \quad \left[\because \quad b^2 = \dfrac{V^2}{n^2} + a^2\right].$

Figure 9



Points to be Noted

For Sum No-22

OA → Initial distance; OP → Distance at any time t.

O → fixed point (considering it as origin).

Law of motion given by

$$m\frac{d^2x}{dt^2} = -mn^2x \quad \text{(−ve sign for attractive force)}$$

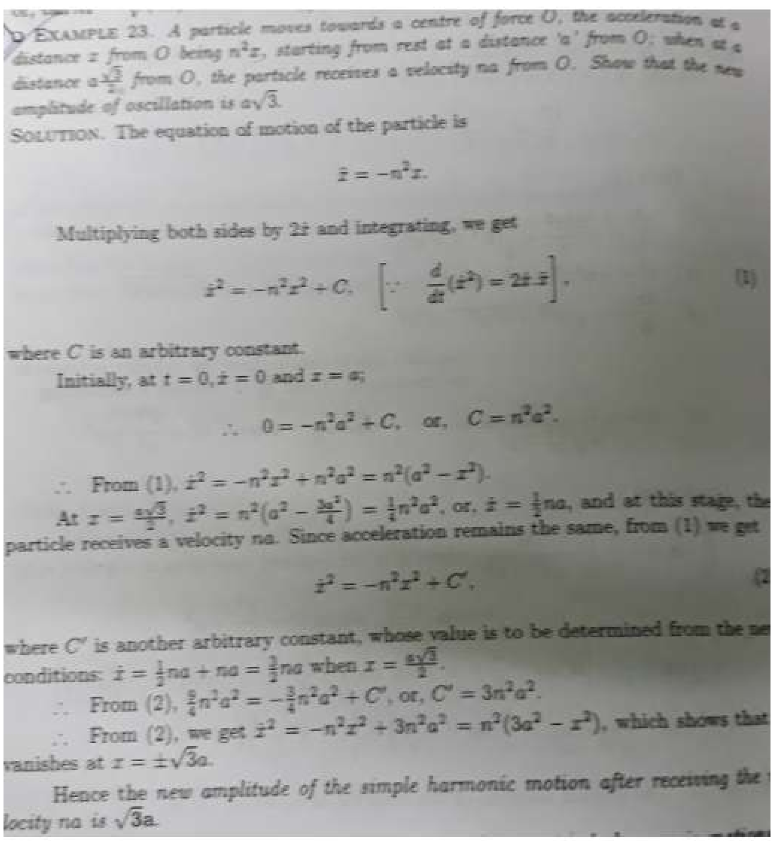
For calculating $\frac{dx}{dt}$, negative sign is taken:-

$$v^2 = \left(\frac{dx}{dt}\right)^2 = -n^2x^2 + V^2 + n^2a^2$$

$$\therefore \frac{dx}{dt} = -\sqrt{(V^2+n^2a^2) - n^2x^2}$$

This negative sign is required as the velocity is in direction towards origin O (∵ force is attractive in nature and particle is moving towards origin) whereas $\frac{dx}{dt}$ is towards x increasing direction.

Figure 10



EXAMPLE 23. *A particle moves towards a centre of force O, the acceleration at a distance x from O being n^2x, starting from rest at a distance 'a' from O; when at a distance $a\frac{\sqrt{3}}{2}$ from O, the particle receives a velocity na from O. Show that the new amplitude of oscillation is $a\sqrt{3}$.*

SOLUTION. The equation of motion of the particle is

$$\ddot{x} = -n^2x.$$

Multiplying both sides by $2\dot{x}$ and integrating, we get

$$\dot{x}^2 = -n^2x^2 + C, \quad \left[\because \frac{d}{dt}(\dot{x}^2) = 2\dot{x}.\ddot{x}\right]. \tag{1}$$

where C is an arbitrary constant.

Initially, at $t = 0, \dot{x} = 0$ and $x = a$;

$$\therefore \quad 0 = -n^2a^2 + C, \quad \text{or,} \quad C = n^2a^2.$$

$\therefore$ From (1), $\dot{x}^2 = -n^2x^2 + n^2a^2 = n^2(a^2 - x^2)$.

At $x = \frac{a\sqrt{3}}{2}$, $\dot{x}^2 = n^2(a^2 - \frac{3a^2}{4}) = \frac{1}{4}n^2a^2$, or, $\dot{x} = \frac{1}{2}na$, and at this stage, the particle receives a velocity na. Since acceleration remains the same, from (1) we get

$$\dot{x}^2 = -n^2x^2 + C', \tag{2}$$

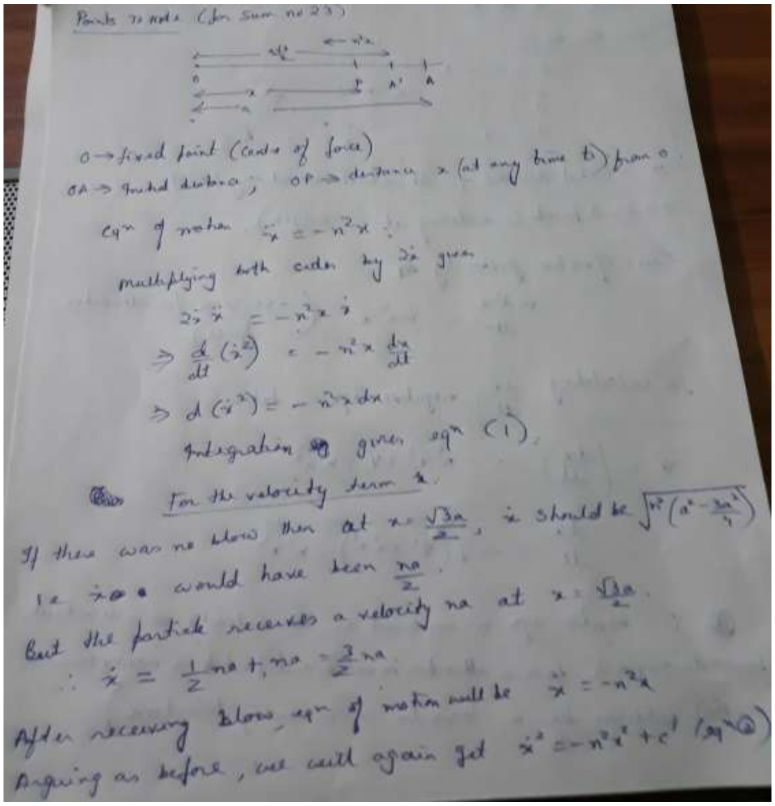
where C' is another arbitrary constant, whose value is to be determined from the ne[w] conditions: $\dot{x} = \frac{1}{2}na + na = \frac{3}{2}na$ when $x = \frac{a\sqrt{3}}{2}$.

$\therefore$ From (2), $\frac{9}{4}n^2a^2 = -\frac{3}{4}n^2a^2 + C'$, or, $C' = 3n^2a^2$.

$\therefore$ From (2), we get $\dot{x}^2 = -n^2x^2 + 3n^2a^2 = n^2(3a^2 - x^2)$, which shows that [$\dot{x}$] vanishes at $x = \pm\sqrt{3}a$.

Hence the *new amplitude of the simple harmonic motion after receiving the* [ve]*locity na is* $\sqrt{3}a$.

Figure 11



Points to note (for Sum no 23)

O → fixed point (Centre of force)

OA → Initial distance a; OP → distance x (at any time t) from O.

eqn of motion $\ddot{x} = -n^2 x$;

multiplying both sides by $2\dot{x}$ gives

$2\dot{x}\ddot{x} = -n^2 x \dot{x}$

$\Rightarrow \frac{d}{dt}(\dot{x}^2) = -n^2 x \frac{dx}{dt}$

$\Rightarrow d(\dot{x}^2) = -n^2 x\, dx$

Integration gives eqn (1).

For the velocity term $\dot{x}$.

If there was no blow then at $x = \frac{\sqrt{3}a}{2}$, $\dot{x}$ should be $\sqrt{n^2\left(a^2 - \frac{3a^2}{4}\right)}$

i.e. $\dot{x}$ would have been $\frac{na}{2}$.

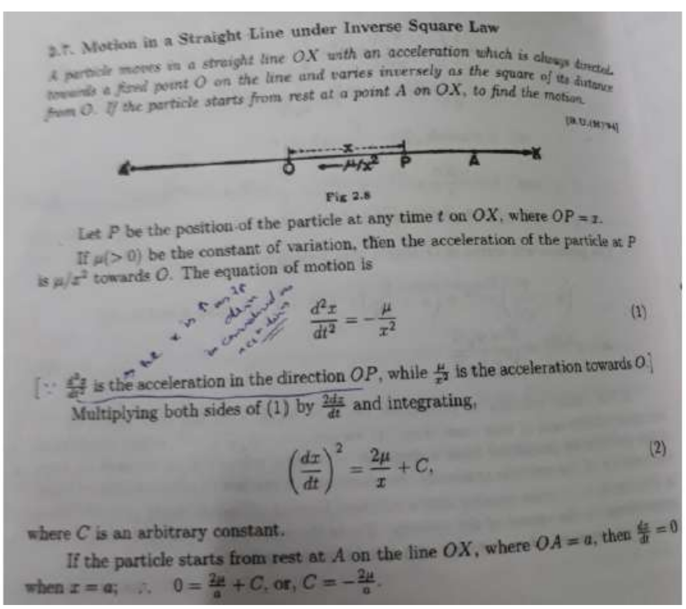
But the particle receives a velocity na at $x = \frac{\sqrt{3}a}{2}$.

$\therefore \dot{x} = \frac{1}{2}na + na = \frac{3}{2}na$

After receiving blow, eqn of motion will be $\ddot{x} = -n^2 x$

Arguing as before, we will again get $\dot{x}^2 = -n^2x^2 + c'$ (eqn (2))

Figure 12



2.7. Motion in a Straight Line under Inverse Square Law

A particle moves in a straight line OX with an acceleration which is always directed towards a fixed point O on the line and varies inversely as the square of its distance from O. If the particle starts from rest at a point A on OX, to find the motion.

[B.U.(H)'94]

Fig 2.8

Let P be the position of the particle at any time t on OX, where $OP = x$.

If $\mu(>0)$ be the constant of variation, then the acceleration of the particle at P is μ/x^2 towards O. The equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2} \quad (1)$$

[$\because \frac{d^2x}{dt^2}$ is the acceleration in the direction OP, while $\frac{\mu}{x^2}$ is the acceleration towards O.]

Multiplying both sides of (1) by $\frac{2dx}{dt}$ and integrating,

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} + C, \quad (2)$$

where C is an arbitrary constant.

If the particle starts from rest at A on the line OX, where $OA = a$, then $\frac{dx}{dt} = 0$ when $x = a$; $\therefore\ 0 = \frac{2\mu}{a} + C$, or, $C = -\frac{2\mu}{a}$.

Figure 13

$\therefore$ From (2),

$$\left(\frac{dx}{dt}\right)^2 = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right)$$

$$\therefore \frac{dx}{dt} = -\sqrt{2\mu\left(\frac{1}{x} - \frac{1}{a}\right)} = -\sqrt{\frac{2\mu}{a}}\sqrt{\frac{a-x}{x}} \qquad (3)$$

[Negative sign is taken on the right side, because the particle moves towards O, while $\dot{x}$ is the velocity along OX.]

$$\text{or,} \quad \sqrt{\frac{2\mu}{a}}\, dt = -\sqrt{\frac{x}{a-x}}\, dx.$$

Integrating:

$$\sqrt{\frac{2\mu}{a}}\, t = -\int \sqrt{\frac{x}{a-x}}\, dx + D,$$

where D is an arbitrary constant of integration.

To evaluate the integral on the right side, let us put $x = a\cos^2\theta$, so that $dx = -2a\cos\theta\sin\theta\, d\theta$, and we have

$$\sqrt{\frac{2\mu}{a}}\, t = \int \frac{\sqrt{a}\cos\theta}{\sqrt{a}\sin\theta} 2a\cos\theta\sin\theta\, d\theta + D = \int 2a\cos^2\theta\, d\theta + D$$

$$= \int a(1+\cos 2\theta)\, d\theta + D = a\left(\theta + \frac{1}{2}\sin 2\theta\right) + D.$$

$$\text{or,} \quad \sqrt{\frac{2\mu}{a}}\, t = a\cos^{-1}\sqrt{\frac{x}{a}} + \sqrt{ax - x^2} + D \qquad (4)$$

Initially, at $t = 0, x = a$; $\therefore$ $0 = a\cos^{-1}1 + 0 + D$, or, $D = 0$.

$\therefore$ From (4),

$$t = \sqrt{\frac{a}{2\mu}}\left[a\cos^{-1}\sqrt{\frac{x}{a}} + \sqrt{ax - x^2}\right]. \qquad (5)$$

Equations (3) and (5) give respectively the velocity and the time at any point P of its path when $x > 0$.

From (3), the velocity at O (i.e., at $x = 0$) is infinite and from (5) the time required to arrive at O is

$$\sqrt{\frac{a}{2\mu}}[a\cos^{-1}0 + 0] = \frac{\pi a}{2}\sqrt{\frac{a}{2\mu}} = \frac{\pi}{2}\cdot\frac{a^{\frac{3}{2}}}{\sqrt{2\mu}}.$$

Next let us consider the motion of the particle for $x < 0$. In this case, the motion will not satisfy equation (1), the equation of motion will be $\ddot{x} = \mu/x^2 (x < 0)$.

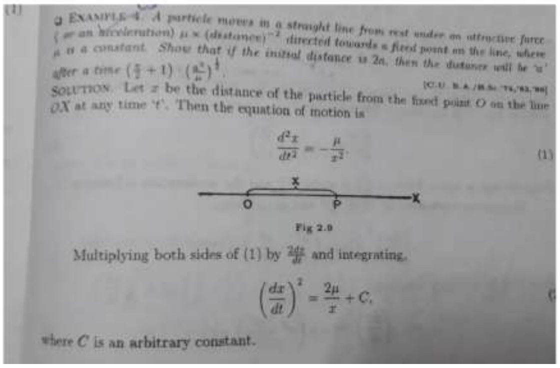
But since on the left side of O on OX acceleration is always directed towards O, the velocity of the particle will be diminished at the same rate as it was increased on the right side of O and, by symmetry, the particle will come to rest at A', where $OA' = OA = a$. Again, at A', the particle has no velocity, but it has an acceleration

Figure 14

towards O and, therefore, it will retrace its path, pass again through O and come to rest at A. Thus the particle will oscillate between A and A', the period of oscillation being

$$4 \times \frac{\pi}{2}\cdot\frac{a^{\frac{3}{2}}}{\sqrt{2\mu}} = 2\pi\cdot\frac{a^{\frac{3}{2}}}{\sqrt{2\mu}}.$$

Figure 15



EXAMPLE 4. *A particle moves in a straight line from rest under an attractive force (or an acceleration) $\mu \times (\text{distance})^{-2}$ directed towards a fixed point on the line, where μ is a constant. Show that if the initial distance is $2a$, then the distance will be 'a' after a time $\left(\frac{\pi}{2}+1\right)\cdot\left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}$.*

[C.U. B.A./B.Sc. '74,'82,'86]

SOLUTION. Let x be the distance of the particle from the fixed point O on the line OX at any time 't'. Then the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}. \tag{1}$$

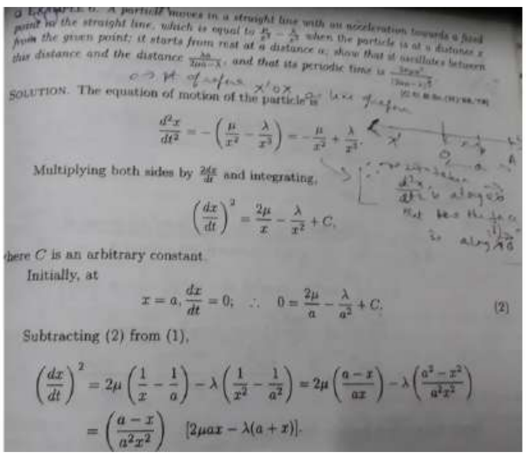
Fig 2.9

Multiplying both sides of (1) by $\frac{2dx}{dt}$ and integrating,

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} + C,$$

where C is an arbitrary constant.

Figure 16



EXAMPLE 6. *A particle moves in a straight line with an acceleration towards a fixed point in the straight line, which is equal to $\frac{\mu}{x^2} - \frac{\lambda}{x^3}$ when the particle is at a distance x from the given point; it starts from rest at a distance a; show that it oscillates between this distance and the distance $\frac{\lambda a}{2\mu a-\lambda}$, and that its periodic time is $\frac{2\pi\mu a^2}{(2\mu a-\lambda)^{\frac{3}{2}}}$.*

SOLUTION. The equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\left(\frac{\mu}{x^2} - \frac{\lambda}{x^3}\right) = -\frac{\mu}{x^2} + \frac{\lambda}{x^3}.$$

Multiplying both sides by $\frac{2dx}{dt}$ and integrating,

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} - \frac{\lambda}{x^2} + C,$$

where C is an arbitrary constant.

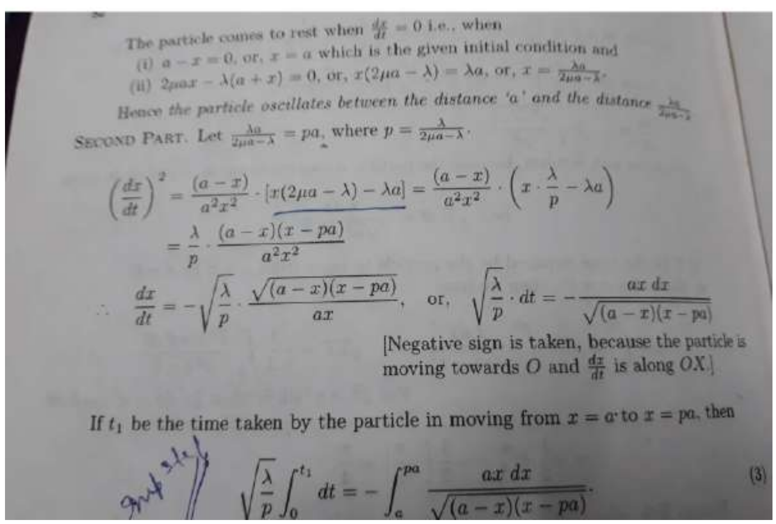
Initially, at

$$x = a, \frac{dx}{dt} = 0; \quad \therefore \quad 0 = \frac{2\mu}{a} - \frac{\lambda}{a^2} + C. \tag{2}$$

Subtracting (2) from (1),

$$\left(\frac{dx}{dt}\right)^2 = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right) - \lambda\left(\frac{1}{x^2} - \frac{1}{a^2}\right) = 2\mu\left(\frac{a-x}{ax}\right) - \lambda\left(\frac{a^2-x^2}{a^2x^2}\right)$$
$$= \left(\frac{a-x}{a^2x^2}\right) \quad [2\mu ax - \lambda(a+x)].$$

Figure 17



The particle comes to rest when $\frac{dx}{dt} = 0$ i.e., when

(i) $a - x = 0$, or, $x = a$ which is the given initial condition and

(ii) $2\mu ax - \lambda(a+x) = 0$, or, $x(2\mu a - \lambda) = \lambda a$, or, $x = \frac{\lambda a}{2\mu a - \lambda}$.

Hence the particle oscillates between the distance 'a' and the distance $\frac{\lambda a}{2\mu a-\lambda}$

SECOND PART. Let $\frac{\lambda a}{2\mu a - \lambda} = pa$, where $p = \frac{\lambda}{2\mu a - \lambda}$.

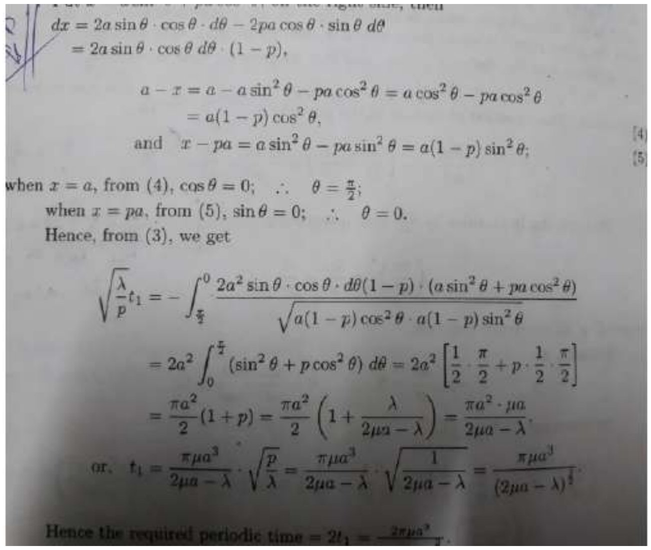
$$\left(\frac{dx}{dt}\right)^2 = \frac{(a-x)}{a^2x^2}\cdot[x(2\mu a - \lambda) - \lambda a] = \frac{(a-x)}{a^2x^2}\cdot\left(x\cdot\frac{\lambda}{p} - \lambda a\right)$$
$$= \frac{\lambda}{p}\cdot\frac{(a-x)(x-pa)}{a^2x^2}$$
$$\therefore \quad \frac{dx}{dt} = -\sqrt{\frac{\lambda}{p}}\cdot\frac{\sqrt{(a-x)(x-pa)}}{ax}, \quad \text{or,} \quad \sqrt{\frac{\lambda}{p}}\cdot dt = -\frac{ax\,dx}{\sqrt{(a-x)(x-pa)}}$$

[Negative sign is taken, because the particle is moving towards O and $\frac{dx}{dt}$ is along OX.]

If t_1 be the time taken by the particle in moving from $x = a$ to $x = pa$, then

$$\sqrt{\frac{\lambda}{p}}\int_0^{t_1} dt = -\int_a^{pa} \frac{ax\,dx}{\sqrt{(a-x)(x-pa)}}. \tag{3}$$

Figure 18



$$dx = 2a\sin\theta\cdot\cos\theta\cdot d\theta - 2pa\cos\theta\cdot\sin\theta\, d\theta$$
$$= 2a\sin\theta\cdot\cos\theta\, d\theta\cdot(1-p),$$

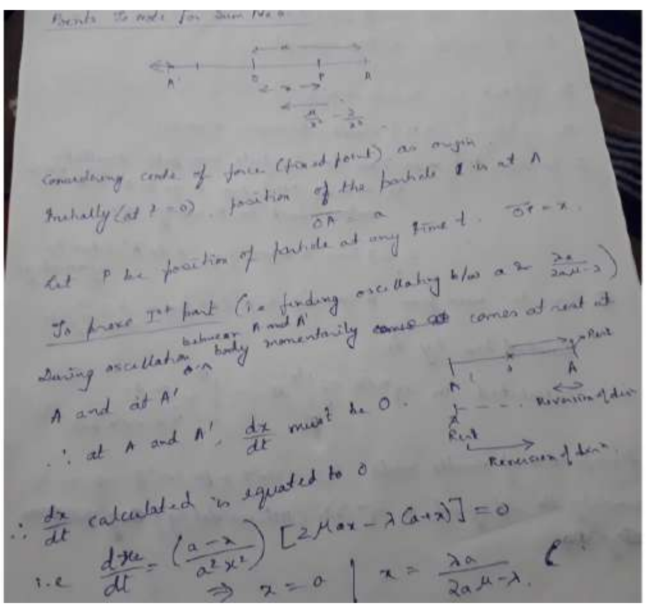
$$a - x = a - a\sin^2\theta - pa\cos^2\theta = a\cos^2\theta - pa\cos^2\theta$$
$$= a(1-p)\cos^2\theta, \quad (4)$$
and $$x - pa = a\sin^2\theta - pa\sin^2\theta = a(1-p)\sin^2\theta; \quad (5)$$

when $x = a$, from (4), $\cos\theta = 0$; $\therefore$ $\theta = \frac{\pi}{2}$;
when $x = pa$, from (5), $\sin\theta = 0$; $\therefore$ $\theta = 0$.
Hence, from (3), we get

$$\sqrt{\frac{\lambda}{p}}t_1 = -\int_{\frac{\pi}{2}}^{0}\frac{2a^2\sin\theta\cdot\cos\theta\cdot d\theta(1-p)\cdot(a\sin^2\theta + pa\cos^2\theta)}{\sqrt{a(1-p)\cos^2\theta\cdot a(1-p)\sin^2\theta}}$$
$$= 2a^2\int_0^{\frac{\pi}{2}}(\sin^2\theta + p\cos^2\theta)\,d\theta = 2a^2\left[\frac{1}{2}\cdot\frac{\pi}{2} + p\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]$$
$$= \frac{\pi a^2}{2}(1+p) = \frac{\pi a^2}{2}\left(1 + \frac{\lambda}{2\mu a - \lambda}\right) = \frac{\pi a^2\cdot\mu a}{2\mu a - \lambda},$$
or, $$t_1 = \frac{\pi\mu a^3}{2\mu a - \lambda}\cdot\sqrt{\frac{p}{\lambda}} = \frac{\pi\mu a^3}{2\mu a - \lambda}\cdot\sqrt{\frac{1}{2\mu a - \lambda}} = \frac{\pi\mu a^3}{(2\mu a - \lambda)^{\frac{3}{2}}}.$$

Hence the required periodic time $= 2t_1 = \frac{2\pi\mu a^3}{\ldots}$.

Figure 19



Points to note for Sum No.

Considering centre of force (fixed point) as origin
Initially (at $t = 0$) position of the particle is at A
$\overline{OA} = a$
Let P be position of particle at any time t. $\overline{OP} = x$.

To prove 1st part (i.e. finding oscillating b/w a & $\frac{\lambda a}{2a\mu - \lambda}$)

During oscillation between A and A' body momentarily ~~comes at~~ comes at rest at A and at A'

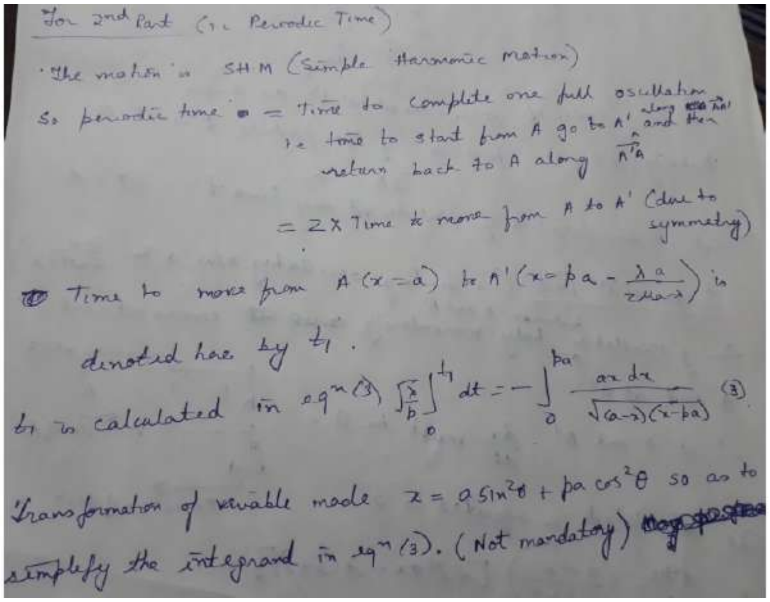
$\therefore$ at A and A', $\frac{dx}{dt}$ must be 0.

$\therefore$ $\frac{dx}{dt}$ calculated is equated to 0

i.e. $\frac{dx}{dt} = \left(\frac{a-x}{a^2x^2}\right)\left[2\mu ax - \lambda(a+x)\right] = 0$

$\Rightarrow x = a \mid x = \frac{\lambda a}{2a\mu - \lambda}$.

Figure 20



For 2nd Part (i.e. Periodic Time)

· The motion is SHM (Simple Harmonic Motion)

So periodic time = Time to complete one full oscillation i.e. time to start from A go to A' along $\overline{AA'}$ and then return back to A along $\overline{A'A}$

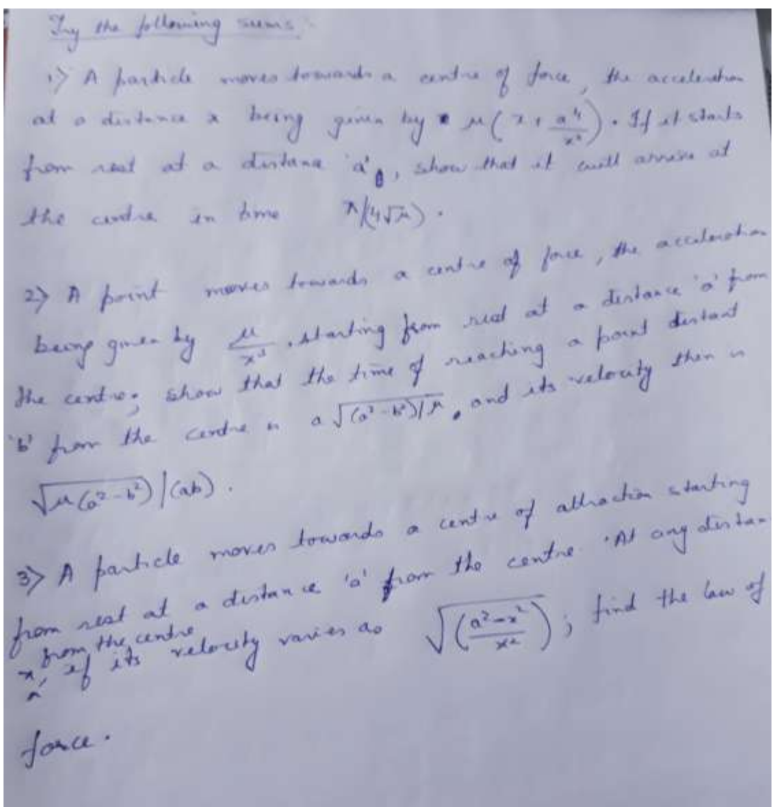
= 2 × Time to move from A to A' (due to symmetry)

Time to move from A $(x = a)$ to A' $\left(x = pa - \frac{\lambda a}{2\mu a - \lambda}\right)$ is denoted here by t_1.

t_1 is calculated in eqn (3) $\sqrt{\frac{\lambda}{b}}\int_0^{t_1} dt = -\int_0^{pa} \frac{ax\,dx}{\sqrt{(a-x)(x-pa)}}$ (3)

Transformation of variable made $x = a\sin^2\theta + pa\cos^2\theta$ so as to simplify the integrand in eqn (3). (Not mandatory)

Figure 21



Try the following sums:

1) A particle moves towards a centre of force, the acceleration at a distance x being given by $\mu\left(x + \frac{a^4}{x^3}\right)$. If it starts from rest at a distance 'a', show that it will arrive at the centre in time $\pi/(4\sqrt{\mu})$.

2) A point moves towards a centre of force, the acceleration being given by $\frac{\mu}{x^3}$, starting from rest at a distance 'a' from the centre. Show that the time of reaching a point distant 'b' from the centre is $a\sqrt{(a^2-b^2)/\mu}$, and its velocity then is $\sqrt{\mu(a^2-b^2)}/(ab)$.

3) A particle moves towards a centre of attraction starting from rest at a distance 'a' from the centre. At any distance x from the centre, if its velocity varies as $\sqrt{\left(\frac{a^2-x^2}{x^2}\right)}$; find the law of force.

Figure 22