SEMESTER-IV

LECTURE NOTES ON

Vector 1st PART

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REFERENCE BOOK: **VECTOR ANALYSIS BY MAITY GHOSH**

8.40 Gradient of a scalar point function

We shall now associate the *directional derivative* of a scalar point function at a point with a special vector called the **gradient**.

DEFINITION. Let S be a level surface of a scalar point function f and let P be a given point on S. Suppose we denote the unit vector normal to S at P by **n** so that **n** points in the direction of increasing f.

(i) The directional derivative of f along **n** is known as the **normal derivative** and is denoted by

$$\frac{df}{dn} \quad \text{or} \quad df/dn \quad (n = \text{distance from } P \text{ measured along } \mathbf{n}).$$

(ii) The gradient of f (denoted by grad f) at the point P is defined to be a vector having magnitude $= df/dn$ and direction same as that of n;

$$\text{i.e., grad } f = \frac{df}{dn}\mathbf{n}.$$

Figure 1

Note. This definition is *invariant* in the sense that grad f is independent of the choice of the basis.

We observe that for all points of a region grad f constitutes a **vector field** determined by a **scalar field** f.

Relation between directional derivative of a scalar point function f at a point in any given direction and gradient of f at that point.

Theorem 8.40.1 *The directional derivative of a scalar field f at a point P and in the direction of a unit vector* **u** *through P is given by*

$$\frac{df}{ds} = \mathbf{u}\cdot \text{grad } f \quad [s \text{ is the distance measured from } P \text{ along } \mathbf{u}] \qquad \text{(II)}$$

that is, *the directional derivative of f along* **u** *is the component of* grad f *in that direction.*

Figure 2

the direction in which the maximum value of the directional derivative of f occurs and the length of grad $f = |\text{grad } f| =$ *maximum rate of change of f.*

2. Let

$$\frac{df}{ds_1}, \ \frac{df}{ds_2}, \ \frac{df}{ds_3}$$

be the directional derivatives, in the directions of *any three mutually orthogonal unit vectors* $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Then since the directional derivatives in any direction is the component of grad f in that directions it follows:

$$\mathbf{e}_1\cdot \text{grad } f = \frac{df}{ds_1}, \quad \mathbf{e}_2\cdot \text{grad } f = \frac{df}{ds_2}, \quad \mathbf{e}_3\cdot \text{grad } f = \frac{df}{ds_3}.$$

Hence in compact form

$$\text{grad } f = \frac{df}{ds_1}\mathbf{e}_1 + \frac{df}{ds_2}\mathbf{e}_2 + \frac{df}{ds_3}\mathbf{e}_3. \qquad \text{(III)}$$

In particular, if we identify $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ with the fundamental system **i**, **j** and **k**, the corresponding directional derivatives are the partial derivatives

$$\frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y} \ \text{and} \ \frac{\partial f}{\partial z}.$$

Accordingly, we have

$$\text{grad } f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}. \qquad \text{(IV)}$$

Figure 3

Example 8.50.3. *If $f = y/(x^2+y^2)$, find the magnitude of the directional derivative making a 30°-angle with the positive x-axis at the point* $(0,1)$.

Solution. $\text{grad } f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \frac{-2xy}{(x^2+y^2)^2}\mathbf{i} + \frac{x^2-y^2}{(x^2+y^2)^2}\mathbf{j}.$

At the point $(0,1)$, grad $f = -\mathbf{j} = (0,-1)$.

The directional derivative is given by

$$\frac{df}{ds} = \mathbf{u}\cdot \text{grad } f, \text{ where } \mathbf{u} = \cos 30°\mathbf{i} + \sin 30°\mathbf{j}$$

$$= \cos 30°(0) - \sin 30°(1) = -\frac{1}{2}.$$

Example 8.50.4. *Find the gradient and the unit vector normal to the level surface $x^2 + y - z = 1$ at the point* $(1,0,0)$.

Solution.

$$f = x^2 + y - z$$

$$\text{grad } f = 2x\mathbf{i} + \mathbf{j} - \mathbf{k} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} \text{ at } (1,0,0).$$

We know that the vector grad f is in the direction of **n** where **n** is the unit vector normal to the surface at $(1,0,0)$.

$$\therefore \quad \mathbf{n} = \frac{\sqrt{6}}{6}(2\mathbf{i} + \mathbf{j} - \mathbf{k}).$$

Example 8.50.5. *Find the directional derivative of $f = xy+yz+zx$ in the direction of the vector* $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ *at* $(1,2,0)$.

Solution. $f = xy + yz + zx$; unit vector along $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ is $\frac{1}{3}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$

$$\text{grad } f = (y+z)\mathbf{i} + (z+x)\mathbf{j} + (x+y)\mathbf{k}$$

$$\frac{df}{ds} = \frac{\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{3}\cdot[(y+z)\mathbf{i} + (z+x)\mathbf{j} + (x+y)\mathbf{k}]$$

$$= \frac{1}{3}(4x+3y+3z), \text{ at the point } (x,y,z)$$

$$= \frac{10}{3}, \text{ at the point } (1,2,0).$$

Figure 4

8.60 The Del (or Nabla) Operator ∇

We defined the gradient of a scalar point function f independent of co-ordinate axes. But if rectangular cartesian system be considered with unit vectors **i**, **j**, **k** along the three mutually perpendicular axes, then

$$\text{grad } f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}.$$

We now introduce a symbol *del* (or *nabla*) ∇ (in rectangular system) as a vector operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

Hence

$$\text{grad } f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)f$$

i.e., ∇f may be considered as the result produced when the vector

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \quad \text{acts on } f;$$

Figure 5

or else ∇f may be assumed to be the multiplication of a vector

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \quad \text{by a scalar } f.$$

Because of its dual characteristics the symbol ∇ is known as **differential operator**.

Again, we have observed before that the directional derivative of direction of the unit vector $\mathbf{u} = \dfrac{df}{ds} = \mathbf{u}\cdot \text{grad} f$, where s is the distanc along $\mathbf{u}$.

We may now write

$$\frac{df}{ds} = \mathbf{u}\cdot \nabla f \quad \textit{or in equivalent form, } (\mathbf{u}\cdot \nabla) f$$

i.e., **the operator d/ds acting on $f \equiv$ the operator $(\mathbf{u}\cdot \nabla)$ actir**

Next consider a vector $\mathbf{s}$ of magnitude s (unit vector along s is $\mathbf{u}$): t Now

$$s\frac{df}{ds} = s(\mathbf{u}\cdot \nabla) f = (s\mathbf{u}\cdot \nabla) f = (\mathbf{s}\cdot \nabla) f$$

i.e., **the operator $s\dfrac{d}{ds}$ acting on $f \equiv$ the operator $(\mathbf{s}\cdot \nabla)$ acting**

Note. Since d/ds may act on a vector point function $\mathbf{F}$ we shall write

$$\frac{d}{ds}\mathbf{F} = (\mathbf{u}\cdot \nabla)\mathbf{F}.$$

But we shall **never write** $\mathbf{u}(\nabla \mathbf{F})$ in this case, because then, $\nabla \mathbf{F}$ becomes meaningless.

Figure 6

8.70 The Operators $\nabla, \nabla\cdot, \nabla\times$ and ∇^2: Gradient, Divergence, Curl and Laplacian [1]

All the functions considered in these discussions will be supposed to be continuou differentiable.

In the previous article we have introduced

$$\text{grad} f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \nabla f.$$

We now introduce three more operators $\nabla\cdot$, $\nabla\times$ and ∇^2:

$$\nabla\cdot \equiv \mathbf{i}\cdot \frac{\partial}{\partial x} + \mathbf{j}\cdot \frac{\partial}{\partial y} + \mathbf{k}\cdot \frac{\partial}{\partial z}$$

$$\nabla\times \equiv \mathbf{i}\times \frac{\partial}{\partial x} + \mathbf{j}\times \frac{\partial}{\partial y} + \mathbf{k}\times \frac{\partial}{\partial z}$$

$$\text{and} \quad \nabla^2 = \nabla\cdot\nabla \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

DO NOT FORGET : ∇ IS AN OPERATOR AND ALSO IT IS A VECTOR.

Figure 7

Let **F** be a vector point function and suppose

$$\mathbf{F} = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$$
$$= F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} \text{ (in shorter form)}$$

Then

$$\nabla\cdot\mathbf{F} = \mathbf{i}\cdot\frac{\partial\mathbf{F}}{\partial x} + \mathbf{j}\cdot\frac{\partial\mathbf{F}}{\partial y} + \mathbf{k}\cdot\frac{\partial\mathbf{F}}{\partial z} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$\left[\because\ \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right.$ and $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$. Take dot produc

$$\nabla\times\mathbf{F} = \mathbf{i}\times\frac{\partial\mathbf{F}}{\partial x} + \mathbf{j}\times\frac{\partial\mathbf{F}}{\partial y} + \mathbf{k}\times\frac{\partial\mathbf{F}}{\partial z}$$
$$= \mathbf{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

and finally if f is a scalar point function, then

Figure 8

Illustrations

Example 8.70.1. *If* $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, *then find* $\nabla\cdot\mathbf{F}$ *and* $\nabla\times\mathbf{F}$.

Solution.

$$\nabla\cdot\mathbf{F} = \sum \mathbf{i}\cdot\frac{\partial\mathbf{F}}{\partial x} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$
$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2(x+y+z)$$

and $$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$
$$= \mathbf{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$
$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

Figure 9

(i) $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}.$

(ii) curl $\mathbf{F}$ (also known as *rot* $\mathbf{F}$) $= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$

2. The operator $\nabla^2 = \nabla\cdot\nabla$ is called the **Laplacian**.

If f is a scalar point function then

$$\nabla^2 f = 0 \quad \text{or,} \quad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is called **Laplace equation.**

Any function f which satisfies the partial differential equation $\nabla^2 u = 0$ is called a **Harmonic function**.

Remember, however, that if F be a vector point function, then $\nabla^2\mathbf{F}$ will mean $\nabla(\nabla\cdot\mathbf{F})$ or (grad div $\mathbf{F}$).

Example 8.70.2. *If* $\mathbf{F} = xy\sin z\,\mathbf{i} + y^2\sin x\,\mathbf{j} + z^2\sin xy\,\mathbf{k}$, *then find* div $\mathbf{F}$ *at the point* $(0, \pi/2, \pi/2)$.

Solution.

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(xy\sin z) + \frac{\partial}{\partial y}(y^2\sin x) + \frac{\partial}{\partial z}(z^2\sin xy) \\ &= y\sin z + 2y\sin x + 2z\sin xy, \text{ at any point } (x, y, z) \\ &= \pi/2 \quad (\text{if we put } x = 0, y = \pi/2, z = \pi/2)\end{aligned}$$

Figure 10

i.e., when $1 + 1 + a = 0$, i.e., when $a = -2$.

Example 8.70.5. *Find the constants* a, b, c *so that the vector*

$$\mathbf{w} = (x + 2y + az)\mathbf{i} + (bx - 3y - z)\mathbf{j} + (4x + cy + 2z)\mathbf{k}$$

becomes irrotational.

Solution. $\mathbf{w}$ is irrotational when $\nabla \times \mathbf{w} = 0$,

$$\text{i.e., when} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$$

i.e., when $\left\{\dfrac{\partial}{\partial y}(4x + cy + 2z) - \dfrac{\partial}{\partial z}(bx - 3y - z)\right\}\mathbf{i} +$ etc. $=$

i.e., when $\mathbf{i}(c + 1) - \mathbf{j}(4 - a) + \mathbf{k}(b - 2) = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$

i.e., when $c = -1, 4 = a, b = 2$.

Example 8.70.6. *If* $\varphi = 2x^3y^2z^4$, *then prove that*

$$\operatorname{div}(\operatorname{grad}\varphi) = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2.$$

Solution. $\operatorname{div}(\operatorname{grad}\varphi) = \nabla\cdot\nabla\varphi = (\nabla\cdot\nabla)\varphi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$

Now $\dfrac{\partial\varphi}{\partial x} = 6x^2y^2z^4$, $\dfrac{\partial^2\varphi}{\partial x^2} = 12xy^2z^4$; etc. Now complete.

Figure 11

Example 8.70.7. *Verify* $\nabla^2 u = 0$ *if* $u = x^2 - y^2 + 4z$. [Similar to E

Problems involving r and r.

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

IMPORTANT RESULTS **(May be used wherever necessary)**

Example 8.70.8. *Given* $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *and* $\mathbf{c}$ *is any constant vecto*

(i) $\operatorname{div}\mathbf{r} = 3$, $\operatorname{div}\mathbf{c} = 0$; (ii) $\operatorname{curl}\mathbf{r} = \mathbf{0}$, $\operatorname{curl}\mathbf{c} = \mathbf{0}$;

(iii) $\operatorname{div}(\mathbf{r} \times \mathbf{c}) = 0$ *but* $\operatorname{curl}(\mathbf{r} \times \mathbf{c}) = -2\mathbf{c}$.

Solution. (i) $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; $\operatorname{div}\mathbf{r} = \nabla \cdot \mathbf{r}$, i.e., $\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) +$;

$$\text{(ii)}\ \operatorname{curl}\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

Figure 12

Similarly, $\operatorname{curl}\mathbf{c} = \mathbf{0}$ (Take, $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$).

$$\text{(iii)}\ \mathbf{r} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{i}(c_3 y - c_2 z) - \mathbf{j}(c_3 x - c_1 z) + \mathbf{k}(c_2 x -$$

Now check $\nabla \cdot \mathbf{r} \times \mathbf{c} = \frac{\partial}{\partial x}(c_3 y - c_2 z) +$ etc. $= 0$.

$$\text{But}\ \operatorname{curl}(\mathbf{r} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ c_3 y - c_2 z & c_1 z - c_3 x & c_2 x - c_1 y \end{vmatrix}$$

$$= -2c_1\mathbf{i} - 2c_2\mathbf{j} - 2c_3\mathbf{k} = -2\mathbf{c}.$$

Figure 13

Differentiation Formulas : Proofs given below

SUMS:

I. $\nabla(f \pm g) = \nabla f \pm \nabla f$

II. $\nabla \cdot (\mathbf{F} \pm \mathbf{G}) = \nabla \cdot \mathbf{F} \pm \nabla \cdot \mathbf{G}$

III. $\nabla \times (\mathbf{F} \pm \mathbf{G}) = \nabla \times \mathbf{F} \pm \nabla \times G$.

Thus **grad, div** or **curl** *is each distributive with respect to the sum and difference of functions* (*scalar point function or vector point function, as the case may be*).

PRODUCTS:

IV. $\nabla(fg) = f\nabla g + g\nabla f$ [$\operatorname{grad}(fg) = f \operatorname{grad} g + g \operatorname{grad} f$]

V. $\nabla(\mathbf{F} \cdot \mathbf{G}) = \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times F) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}$

VI. $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl}\mathbf{F} - \mathbf{F} \cdot \operatorname{curl}\mathbf{G} \cdot$

i.e., $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

VII. $\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

$\operatorname{curl}\mathbf{F} \times \mathbf{G} = \mathbf{F} \operatorname{div}\mathbf{G} - \mathbf{G} \operatorname{div}\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G}$

Figure 14

Divergence and Curl of one scalar function multiplied by one vector function:

VIII. $\nabla\cdot(f\mathbf{F}) = f(\nabla\cdot\mathbf{F}) + \mathbf{F}\cdot\nabla f$
divergence of $f\mathbf{F} = f$ divergence of $\mathbf{F} + \mathbf{F}\cdot$ grad f.

IX. $\nabla\times(f\mathbf{F}) = \nabla f\times\mathbf{F} + f(\nabla\times\mathbf{F})$
curl of $f\mathbf{F} =$ grad $f\times\mathbf{F} + f$ curl $\mathbf{F}$. [C.H. 2000]

Second order differential operators:

X. $\nabla\cdot\nabla f = \nabla^2 f$ (div of grad $f \equiv$ Laplacian f where f is a scalar point function)

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

But $\nabla^2\mathbf{F} = \nabla(\nabla\cdot\mathbf{F}) =$ grad (div $\mathbf{F}$)

XI. $\nabla\times\nabla f = \mathbf{0}$; $\nabla\cdot(\nabla\times\mathbf{F}) = 0$.
curl (grad f) $= \mathbf{0}$; div (curl $\mathbf{F}$) $= 0$.

XII. $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \mathbf{F}\nabla^2\mathbf{F}$
curl curl $\mathbf{F} =$ grad (div $\mathbf{F}$) $-$ Laplacian $\mathbf{F}$

Proofs of differentiation Formulas:

Formula I. To prove

$$\nabla(f+g) = \sum \hat{\imath}\frac{\partial}{\partial x}(f+g) = \sum \hat{\imath}\left\{\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right\}$$
$$= \sum \hat{\imath}\frac{\partial f}{\partial x} + \sum \hat{\imath}\frac{\partial g}{\partial x} = \nabla f + \nabla g$$

Similar proof for $\nabla(f-g) = \nabla f - \nabla g$.

Figure 15

Formula II. To prove: $\nabla\cdot(\mathbf{F}+\mathbf{G}) = \nabla\cdot\mathbf{F} + \nabla\cdot\mathbf{G}$.

$$\begin{aligned}
\nabla\cdot(\mathbf{F}+\mathbf{G}) &= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\mathbf{F}+\mathbf{G}) \\
&= \hat{\imath}\cdot\frac{\partial}{\partial x}(\mathbf{F}+\mathbf{G}) + \hat{\jmath}\cdot\frac{\partial}{\partial y}(\mathbf{F}+\mathbf{G}) + \hat{k}\cdot\frac{\partial}{\partial z}(\mathbf{F}+\mathbf{G}) \\
&= \hat{\imath}\cdot\left(\frac{\partial\mathbf{F}}{\partial x} + \frac{\partial\mathbf{G}}{\partial x}\right) + \hat{\jmath}\cdot\left(\frac{\partial\mathbf{F}}{\partial y} + \frac{\partial\mathbf{G}}{\partial y}\right) + \hat{k}\cdot\left(\frac{\partial\mathbf{F}}{\partial z} + \frac{\partial\mathbf{G}}{\partial z}\right) \\
&= \left(\hat{\imath}\cdot\frac{\partial\mathbf{F}}{\partial x} + \hat{\jmath}\cdot\frac{\partial\mathbf{F}}{\partial y} + \hat{k}\cdot\frac{\partial\mathbf{F}}{\partial z}\right) + \left(\hat{\imath}\cdot\frac{\partial\mathbf{G}}{\partial x} + \hat{\jmath}\cdot\frac{\partial\mathbf{G}}{\partial y} + \hat{k}\cdot\frac{\partial\mathbf{G}}{\partial z}\right) \\
&= \nabla\cdot\mathbf{F} + \nabla\cdot\mathbf{G}
\end{aligned}$$

Formula III. To prove: $\nabla\times(\mathbf{F}+\mathbf{G}) = \nabla\times\mathbf{F} + \nabla\times\mathbf{G}$.

$$\begin{aligned}
\nabla\times(\mathbf{F}+\mathbf{G}) &= \mathbf{i}\times\frac{\partial}{\partial x}(\mathbf{F}+\mathbf{G}) + \mathbf{j}\times\frac{\partial}{\partial y}(\mathbf{F}+\mathbf{G}) + \mathbf{k}\times\frac{\partial}{\partial z}(\mathbf{F}+\mathbf{G}) \\
&= \sum \mathbf{i}\times\frac{\partial}{\partial x}(\mathbf{F}+\mathbf{G}) = \sum \mathbf{i}\times\left\{\frac{\partial\mathbf{F}}{\partial x} + \frac{\partial\mathbf{G}}{\partial x}\right\} \\
&= \sum \mathbf{i}\times\frac{\partial\mathbf{F}}{\partial x} + \sum \mathbf{i}\times\frac{\partial\mathbf{G}}{\partial x} \\
&= \nabla\times\mathbf{F} + \nabla\times\mathbf{G}.
\end{aligned}$$

Figure 16

Formula IV. To prove: $\nabla(fg) = f\nabla g + g\nabla f$.

$$\begin{aligned}\nabla(fg) &= \mathbf{i}\frac{\partial}{\partial x}(fg) + \mathbf{j}\frac{\partial}{\partial y}(fg) + \mathbf{k}\frac{\partial}{\partial z}(fg)\\ &= \sum \mathbf{i}\left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right) = \sum \mathbf{i}f\frac{\partial g}{\partial x} + \sum \mathbf{i}g\frac{\partial f}{\partial x}\\ &= f\sum \mathbf{i}\frac{\partial g}{\partial x} + g\sum \mathbf{i}\frac{\partial f}{\partial x} = f\nabla g + g\nabla f.\end{aligned}$$

Formula V. To prove:

$$\nabla(\mathbf{F}\cdot\mathbf{G}) = \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}) + (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F}$$

$$\begin{aligned}\nabla(\mathbf{F}\cdot\mathbf{G}) &= \sum \mathbf{i}\frac{\partial}{\partial x}(\mathbf{F}\cdot\mathbf{G}) = \sum \mathbf{i}\left\{\mathbf{F}\cdot\frac{\partial \mathbf{G}}{\partial x} + \mathbf{G}\cdot\frac{\partial \mathbf{F}}{\partial x}\right\}\\ &= \sum \mathbf{i}\mathbf{F}\cdot\frac{\partial \mathbf{G}}{\partial x} + \sum \mathbf{i}\mathbf{G}\cdot\frac{\partial \mathbf{F}}{\partial x}\end{aligned}$$

Now see that

$$\mathbf{F}\times\left(\mathbf{i}\times\frac{\partial \mathbf{G}}{\partial x}\right) = \left(\mathbf{F}\cdot\frac{\partial \mathbf{G}}{\partial x}\right)\mathbf{i} - (\mathbf{F}\cdot\mathbf{i})\frac{\partial \mathbf{G}}{\partial x}$$

Figure 17

Similarly,

$$\mathbf{i}\left(\mathbf{G}\cdot\frac{\partial \mathbf{F}}{\partial x}\right) = \mathbf{G}\times\left(\mathbf{i}\times\frac{\partial \mathbf{F}}{\partial x}\right) + (\mathbf{G}\cdot\mathbf{i})\frac{\partial \mathbf{F}}{\partial x}.$$

Hence,

$$\begin{aligned}\nabla(\mathbf{F}\cdot\mathbf{G}) &= \sum \mathbf{F}\times\left(\mathbf{i}\times\frac{\partial \mathbf{G}}{\partial x}\right) + \sum \mathbf{G}\times\left(\mathbf{i}\times\frac{\partial \mathbf{F}}{\partial x}\right)\\ &\quad + \sum(\mathbf{F}\cdot\mathbf{i})\frac{\partial \mathbf{G}}{\partial x} + \sum(\mathbf{G}\cdot\mathbf{i})\frac{\partial \mathbf{F}}{\partial x}\\ &= \mathbf{F}\times\sum \mathbf{i}\times\frac{\partial \mathbf{G}}{\partial x} + \mathbf{G}\times\sum \mathbf{i}\times\frac{\partial \mathbf{F}}{\partial x}\\ &\quad + \sum(\mathbf{F}\cdot\mathbf{i})\frac{\partial \mathbf{G}}{\partial x} + \sum(\mathbf{G}\cdot\mathbf{i})\frac{\partial \mathbf{F}}{\partial x}\\ &= \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}) + (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F}.\end{aligned}$$

Figure 18

Formula VI. To prove: $\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G})$

$$\begin{aligned}\nabla\cdot(\mathbf{F}\times\mathbf{G}) &= \sum \mathbf{i}\cdot\frac{\partial}{\partial x}(\mathbf{F}\times\mathbf{G}) = \sum \mathbf{i}\cdot\left\{\frac{\partial \mathbf{F}}{\partial x}\times\mathbf{G} + \mathbf{F}\times\frac{\partial \mathbf{G}}{\partial x}\right\}\\ &= \sum \mathbf{i}\cdot\frac{\partial \mathbf{F}}{\partial x}\times\mathbf{G} + \sum \mathbf{i}\cdot\mathbf{F}\times\frac{\partial \mathbf{G}}{\partial x}\\ &= \sum\left(\mathbf{i}\times\frac{\partial \mathbf{F}}{\partial x}\right)\cdot\mathbf{G} - \sum\left(\mathbf{i}\times\frac{\partial \mathbf{G}}{\partial x}\right)\cdot\mathbf{F}\\ &\qquad [\textbf{Remember:}\quad \mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{a}\times\mathbf{b}\cdot\mathbf{c}]\\ &= \nabla\times\mathbf{F}\cdot\mathbf{G} - \nabla\times\mathbf{G}\cdot\mathbf{F} = \mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G}).\end{aligned}$$

Formula VII. To prove:
$\nabla\times(\mathbf{F}\times\mathbf{G}) = \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F}) + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$.

$$\begin{aligned}\nabla\times(\mathbf{F}\times\mathbf{G}) &= \sum \mathbf{i}\times\frac{\partial}{\partial x}(\mathbf{F}\times\mathbf{G}) = \sum \mathbf{i}\times\left\{\frac{\partial \mathbf{F}}{\partial x}\times\mathbf{G} + \mathbf{F}\times\frac{\partial \mathbf{G}}{\partial x}\right\}\\ &= \sum \mathbf{i}\times\left(\mathbf{F}\times\frac{\partial \mathbf{G}}{\partial x}\right) + \sum \mathbf{i}\times\left(\frac{\partial \mathbf{F}}{\partial x}\times\mathbf{G}\right)\\ &= \sum\left\{\left(\mathbf{i}\cdot\frac{\partial \mathbf{G}}{\partial x}\right)\mathbf{F} - (\mathbf{i}\cdot\mathbf{F})\frac{\partial \mathbf{G}}{\partial x}\right\} + \sum\left\{(\mathbf{i}\cdot\mathbf{G})\frac{\partial \mathbf{F}}{\partial x} - \left(\mathbf{i}\cdot\frac{\partial \mathbf{F}}{\partial x}\right)\mathbf{G}\right\}\end{aligned}$$

Arranging in a proper way,

$$\begin{aligned}&= \mathbf{F}\sum \mathbf{i}\cdot\frac{\partial \mathbf{G}}{\partial x} - \mathbf{G}\sum \mathbf{i}\cdot\frac{\partial \mathbf{F}}{\partial x} + \sum(\mathbf{G}\cdot\mathbf{i})\frac{\partial \mathbf{F}}{\partial x} - \sum(\mathbf{F}\cdot\mathbf{i})\frac{\partial \mathbf{G}}{\partial x}\\ &= \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F}) + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}\\ &\qquad\left[\text{see that } (\mathbf{G}\cdot\mathbf{i})\frac{\partial \mathbf{F}}{\partial x} = \left(\mathbf{G}\cdot\mathbf{i}\frac{\partial}{\partial x}\right)\mathbf{F}\right]\end{aligned}$$

Formula VIII can be proved in a similar manner as we have proved **Formula IX** below:

Figure 19

Formula IX. To prove: $\nabla\times(f\mathbf{F}) = \nabla f\times\mathbf{F} + f(\nabla\times\mathbf{F})$

$$\begin{aligned}\nabla\times(f\mathbf{F}) &= \sum \mathbf{i}\times\frac{\partial}{\partial x}(f\mathbf{F}) = \sum \mathbf{i}\times\left\{\frac{\partial f}{\partial x}\mathbf{F} + f\frac{\partial \mathbf{F}}{\partial x}\right\}\\ &= \sum \mathbf{i}\times\frac{\partial f}{\partial x}\mathbf{F} + \sum \mathbf{i}\times f\frac{\partial \mathbf{F}}{\partial x}\\ &= \sum\left\{\mathbf{i}\frac{\partial f}{\partial x}\times\mathbf{F}\right\} + \sum\left(\mathbf{i}\times\frac{\partial \mathbf{F}}{\partial x}\right)f\\ &= \nabla f\times\mathbf{F} + f(\nabla\times\mathbf{F})\end{aligned}$$

In particular, $\nabla\times(f(r)\mathbf{r}) = \mathbf{0}\quad(\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k})$
For,

$$\begin{aligned}\nabla\times(f(r))\mathbf{r} &= \nabla f(r)\times\mathbf{r} + f(r)\nabla\times\mathbf{r}\\ &= \nabla f(r)\times\mathbf{r}\quad(\because\ \nabla\times\mathbf{r} = 0)\\ &= \left\{\frac{\partial}{\partial x}f(r)\hat{\imath} + \frac{\partial}{\partial y}f(r)\hat{\jmath} + \frac{\partial}{\partial z}f(r)\hat{k}\right\}\times\mathbf{r}\\ &= \left\{f'(r)\frac{x}{r}\hat{\imath} + f'(r)\frac{y}{r}\hat{\jmath} + f'(r)\frac{z}{r}\hat{k}\right\}\times\mathbf{r}\\ &= \frac{f'(r)}{r}(x\hat{\imath} + y\hat{\jmath} + z\hat{k})\times\mathbf{r} = \frac{f'(r)}{r}\mathbf{r}\times\mathbf{r} = \mathbf{0}.\end{aligned}$$

In other words, the vector $f(r)\mathbf{r}$ is **irrotational**.

Figure 20