SEMESTER-IV

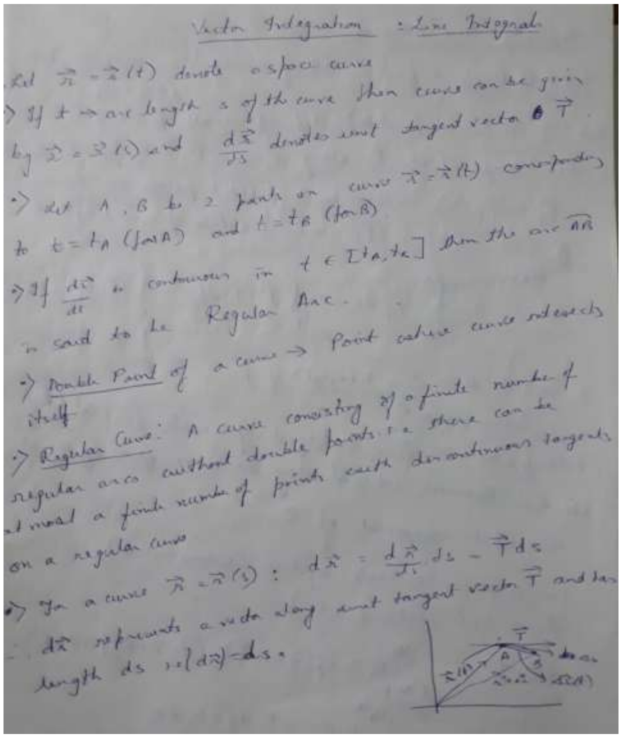
LECTURE NOTES ON

Vector 2^{nd} PART

KAUSHIKI MUKHERJEE

Department of Mathematics, Government Girls' General Degree College, Ekbalpore, Kolkata-700023

REFERENCE BOOK: **VECTOR ANALYSIS BY MAITY GHOSH**



Vector Integration : Line Integral

Let $\vec{r} = \vec{r}(t)$ denote a space curve

→ If t → arc length s of the curve then curve can be given by $\vec{r} = \vec{r}(s)$ and $\frac{d\vec{r}}{ds}$ denotes unit tangent vector $\vec{T}$.

→ Let A, B be 2 points on curve $\vec{r} = \vec{r}(t)$ corresponding to $t = t_A$ (for A) and $t = t_B$ (for B)

→ If $\frac{d\vec{r}}{dt}$ is continuous in $t \in [t_A, t_B]$ then the arc $\overline{AB}$ is said to be Regular Arc.

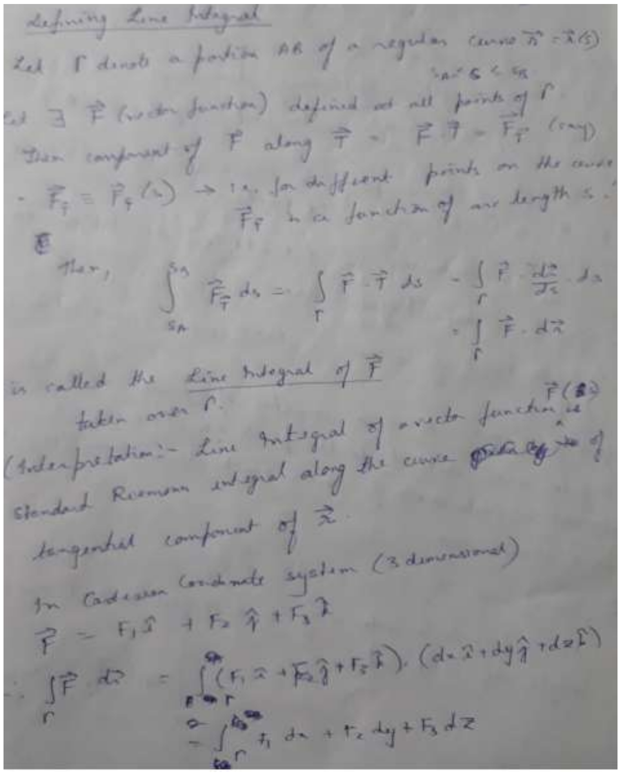
→ Double Point of a curve → Point where curve intersects itself

→ Regular Curve: A curve consisting of a finite number of regular arcs without double points. i.e there can be at most a finite number of points with discontinuous tangents on a regular curve

→ For a curve $\vec{r} = \vec{r}(s)$: $d\vec{r} = \frac{d\vec{r}}{ds} ds = \vec{T} ds$

$d\vec{r}$ represents a vector along unit tangent vector $\vec{T}$ and has length ds i.e $|d\vec{r}| = ds$.

Figure 1



Defining Line Integral

Let Γ denote a portion AB of a regular curve $\vec{r} = \vec{r}(s)$ $s_A \leq s \leq s_B$

Let $\exists$ $\vec{F}$ (vector function) defined at all points of Γ

Then component of $\vec{F}$ along $\vec{T}$ = $\vec{F} \cdot \vec{T} = \vec{F_T}$ (say)

$\therefore \vec{F_T} = \vec{F_T}(s)$ → i.e. for different points on the curve $\vec{F_T}$ is a function of arc length s.

Then,

$$\int_{s_A}^{s_B} \vec{F_T}\, ds = \int_\Gamma \vec{F} \cdot \vec{T}\, ds = \int_\Gamma \vec{F} \cdot \frac{d\vec{r}}{ds}\, ds = \int_\Gamma \vec{F} \cdot d\vec{r}$$

is called the Line Integral of $\vec{F}$ taken over Γ.

(Interpretation:- Line Integral of a vector function $\vec{F}(s)$ is standard Riemann integral along the curve of tangential component of $\vec{r}$.

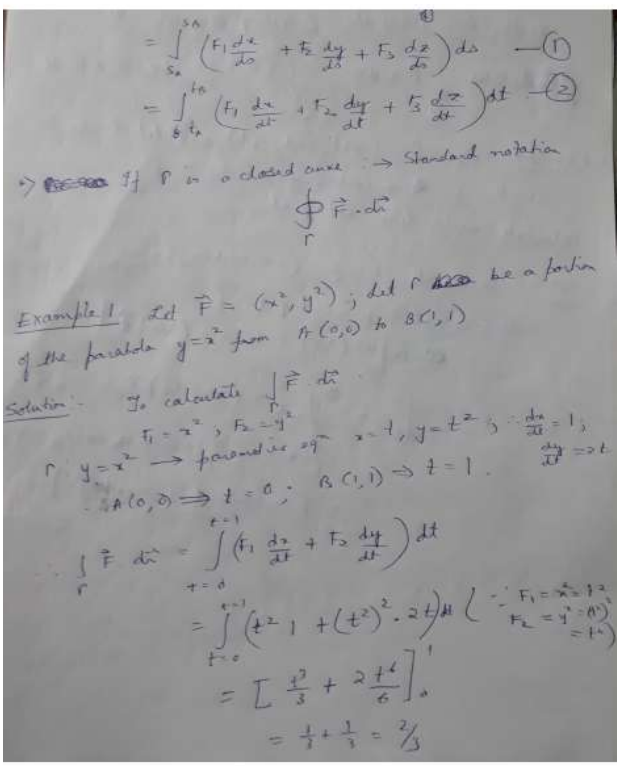
In Cartesian Coordinate system (3 dimensional)

$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

$$\therefore \int_\Gamma \vec{F} \cdot d\vec{r} = \int_\Gamma (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}) \cdot (dx\, \hat{i} + dy\, \hat{j} + dz\, \hat{k})$$

$$= \int_\Gamma F_1\, dx + F_2\, dy + F_3\, dz$$

Figure 2



$$= \int_{s_A}^{s_B} \left(F_1 \frac{dx}{ds} + F_2 \frac{dy}{ds} + F_3 \frac{dz}{ds} \right) ds \quad \text{— (1)}$$

$$= \int_{t_A}^{t_B} \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt \quad \text{— (2)}$$

→ If Γ is a closed curve → Standard notation

$$\oint_\Gamma \vec{F} \cdot d\vec{r}$$

Example 1 Let $\vec{F} = (x^2, y^2)$; Let Γ be a portion of the parabola $y = x^2$ from $A(0,0)$ to $B(1,1)$

Solution:- To calculate $\int_\Gamma \vec{F} \cdot d\vec{r}$

$F_1 = x^2$, $F_2 = y^2$

Γ: $y = x^2$ → parametric eqn $x = t$, $y = t^2$; $\therefore \frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2t$

$\therefore A(0,0) \Rightarrow t = 0$; $B(1,1) \Rightarrow t = 1$.

$$\therefore \int_\Gamma \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=1} \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} \right) dt$$

$$= \int_{t=0}^{t=1} \left(t^2 \cdot 1 + (t^2)^2 \cdot 2t \right) dt \quad \left(\because F_1 = x^2 = t^2,\ F_2 = y^2 = (t^2)^2 = t^4 \right)$$

$$= \left[\frac{t^3}{3} + 2\frac{t^6}{6} \right]_0^1$$

$$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Figure 3

Vector Line Integrals (Working Formula)

$$\int_\Gamma \vec{F} \times d\vec{r} = \int_\Gamma \left(\vec{F} \times \frac{d\vec{r}}{ds}\right) ds = \int_\Gamma (\vec{F} \times \vec{T})\, ds$$

Example Let $\vec{F} = (\cos t, \sin t, z^t)$ and let Γ be the circle $x = a\cos t$, $y = a\sin t$, $z = 0$

Calculate vector line integral.

Sol^n For circle $t = \frac{s}{a}$ ($a \to$ radius, $s \to$ arc length)

$$\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j}$$
$$= a\left[\cos(s/a)\,\hat{i} + \sin(s/a)\,\hat{j}\right]$$

Now $$\vec{T} = \frac{d\vec{r}}{ds} = \frac{d}{ds}\left[a\left(\cos(s/a)\,\hat{i} + \sin(s/a)\,\hat{j}\right)\right]$$
$$= \left[-\sin(s/a)\,\hat{i} + \cos(s/a)\,\hat{j}\right]$$
$$= \left[-\sin t\,\hat{i} + \cos t\,\hat{j}\right]$$

Figure 4

9.30 Surface Integrals

1. Concept of double integral

Let $f(x, y)$ be a real-valued function of two independent variables x and y defined over a finite region S of the xy-plane. Divide the region S into sub-regions of areas $\Delta S_1, \Delta S_2, \cdots, \Delta S_n$ by taking two families of intersecting curves and choose any point (ξ_r, η_r) inside or on the sub-region ΔS_r. Then form the sum

$$S_n = \sum_{r=1}^{n} f(\xi_r, \eta_r)\Delta S_r.$$

If $\lim_{n\to\infty} S_n$ exists and if that is independent of the mode of subdivision of S and the choice of (ξ_r, η_r), then this limit is called the double integral of $f(x, y)$ over the region S and is denoted by

$$\int_S f(x, y)\, dS$$

Remember. If $f(x, y)$ is continuous over S then the double integral exists.

Vector double integral.

If, now, a vector function $\mathbf{F}(x, y)$ is defined at each point (x, y) of the region S then we form as above the vector sum

$$\mathbf{S}_n = \sum_{r=1}^{n} \mathbf{F}(\xi_r, \eta_r)\Delta S_r.$$

If $\lim_{n\to\infty} \mathbf{S}_n$ exists and if the limit is independent of the mode of subdivision and choice of (ξ_r, η_r) points, then that limit is called the **vector double integral** of $\mathbf{F}(x, y)$ over the region S and is denoted by

$$\int_S \mathbf{F}(x, y)\, dS.$$

Figure 5

2. A few remarks about Surfaces and Normals

Let $\mathbf{r}(u, v)$ be a *vector function* of two independent *scalar variables* u and v. To each pair of values of u and v there corresponds a vector whose one end is the origin and the other end is a point in space. As u and v vary this terminal point traces out a surface in space.

Now consider all points on the surface for which v has the constant value v_1 lie on a curve $\mathbf{r}(u, v_1)$ with parameter u. We call this curve a u-curve (u-parameter curve). Similarly we define a v-cure $\mathbf{r}(u_1, v)$ (where u has the constant value, say, u_1 and v is a parameter). These curves are called **parametric curves.** The vector $\mathbf{r}_u = \partial \mathbf{r}/\partial u$ is a tangent vector to the parametric curve v = constant while $\mathbf{r}_v = \partial \mathbf{r}/\partial v$ represents a tangent vector to the parametric curve u = constant. The equations $u = u(t)$, $v = v(t)$ determine the curve $\mathbf{r}\{u(t), v(t)\}$ with parameter t on the surface, the direction of the tangent being given by

$$\frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{dt} = \mathbf{r}_u \frac{du}{dt} + \mathbf{r}_v \frac{dv}{dt},$$

i.e., at a point the **tangent vectors** to all curves on the surface passing through the point lie on a plane formed by $\mathbf{r}_u$ and $\mathbf{r}_v$. This plane is called the **tangent plane** to the surface at that point. The **normal** at a point on a surface is then defined to be a vector perpendicular to the *tangent plane* and hence in the direction of $\mathbf{r}_u \times \mathbf{r}_v$. We choose the positive direction of the unit surface normal $\mathbf{n}$ so that

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|},$$

Figure 6

... unit surface normal $\mathbf{n}$ so that

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|},$$

the vectors $\mathbf{r}_u, \mathbf{r}_v$ and $\mathbf{n}$ form a *riht-handed system*.

(i) The distance ds between two points $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$ on the surface $\mathbf{r}(u, v)$ is given by

$$\begin{aligned} ds^2 &= d\mathbf{r}.d\mathbf{r} = (\mathbf{r}_u du + \mathbf{r}_v dv).(\mathbf{r}_u du + \mathbf{r}_v dv) \\ &= E du^2 + 2F du\, dv + G dv^2, \end{aligned}$$

where $E = \mathbf{r}_u.\mathbf{r}_u$, $F = \mathbf{r}_u.\mathbf{r}_v$ and $G = \mathbf{r}_v.\mathbf{r}_v$.

In particular, $d_u s$ = distance along the u-curve (where $dv = 0$) $= \sqrt{E} du = |\mathbf{r}_u| du$
and $d_v s$ = distance along the v-curve (where $du = 0$) $= \sqrt{G} du = |\mathbf{r}_v| dv$.

(ii) Suppose that a surface S is divided into several sub-regions by means of u-curves and v-curves. If A, B, C, D be four neighbouring points with respective parameter values $(u, v), (u + du, v), (u + du, v + dv)$ and $(u, v + dv)$, then *the scalar element of area dS contained by ABCD* is given by

$$\begin{aligned} dS &= d_u s d_v s \sin w \\ &= \sqrt{EG} du\, dv \frac{|\mathbf{r}_u \times \mathbf{r}_v|}{|\mathbf{r}_u|\, |\mathbf{r}_v|} \\ &= |\mathbf{r}_u \times \mathbf{r}_v| du\, dv; \end{aligned}$$

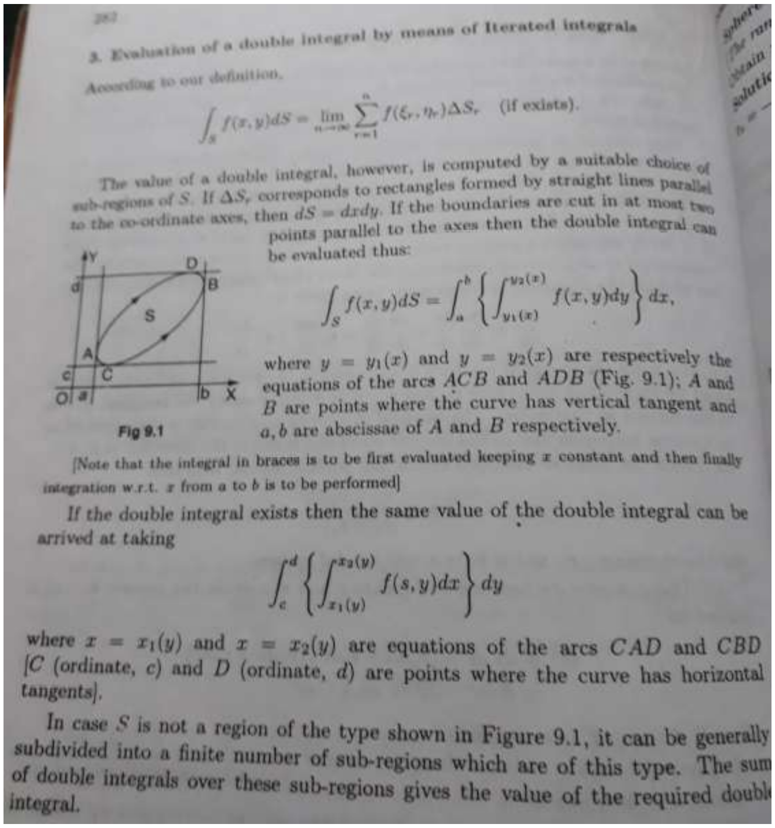
[w is the angle between the tangents to the parametric curves through the point $A(u, v)$]

and then *vector element of area dS* is defined by

$$d\mathbf{S} = \mathbf{r}_u \times \mathbf{r}_v du\, dv = \mathbf{n}|\mathbf{r}_u \times \mathbf{r}_v| du\, dv = \mathbf{n} dS.$$

(iii) If every u-curve is orthogonal to every v-curve then the parametric curves form an **orthogonal system**, the condition for which is $\mathbf{r}_u.\mathbf{r}_v = 0$, i.e., $F = 0$.

Figure 7



3. Evaluation of a double integral by means of Iterated integrals

According to our definition,

$$\int_S f(x,y)dS = \lim_{n\to\infty} \sum_{r=1}^{n} f(\xi_r, \eta_r)\Delta S_r \quad \text{(if exists)}.$$

The value of a double integral, however, is computed by a suitable choice of sub-regions of S. If ΔS_r corresponds to rectangles formed by straight lines parallel to the co-ordinate axes, then $dS = dxdy$. If the boundaries are cut in at most two points parallel to the axes then the double integral can be evaluated thus:

Fig 9.1

$$\int_S f(x,y)dS = \int_a^b \left\{ \int_{y_1(x)}^{y_2(x)} f(x,y)dy \right\} dx,$$

where $y = y_1(x)$ and $y = y_2(x)$ are respectively the equations of the arcs ACB and ADB (Fig. 9.1); A and B are points where the curve has vertical tangent and a, b are abscissae of A and B respectively.

[Note that the integral in braces is to be first evaluated keeping x constant and then finally integration w.r.t. x from a to b is to be performed]

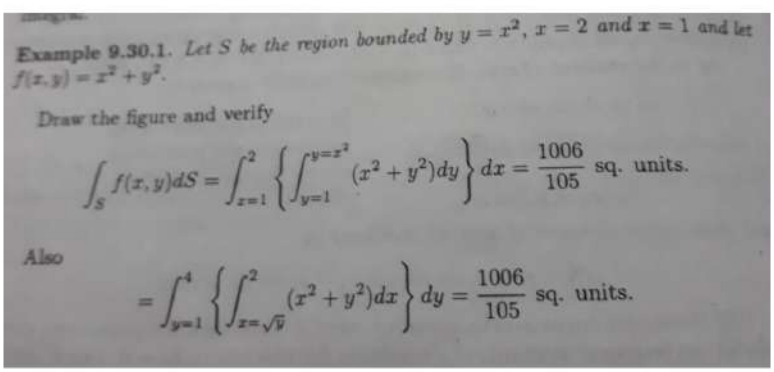
If the double integral exists then the same value of the double integral can be arrived at taking

$$\int_c^d \left\{ \int_{x_1(y)}^{x_2(y)} f(s,y)dx \right\} dy$$

where $x = x_1(y)$ and $x = x_2(y)$ are equations of the arcs CAD and CBD [C (ordinate, c) and D (ordinate, d) are points where the curve has horizontal tangents].

In case S is not a region of the type shown in Figure 9.1, it can be generally subdivided into a finite number of sub-regions which are of this type. The sum of double integrals over these sub-regions gives the value of the required double integral.

Figure 8



Example 9.30.1. *Let S be the region bounded by $y = x^2$, $x = 2$ and $x = 1$ and let $f(x,y) = x^2 + y^2$.*

Draw the figure and verify

$$\int_S f(x,y)dS = \int_{x=1}^{2} \left\{ \int_{y=1}^{y=x^2} (x^2+y^2)dy \right\} dx = \frac{1006}{105} \text{ sq. units.}$$

Also

$$= \int_{y=1}^{4} \left\{ \int_{x=\sqrt{y}}^{2} (x^2+y^2)dx \right\} dy = \frac{1006}{105} \text{ sq. units.}$$

Figure 9

Example 9.30.2. *Let the surface $\mathbf{r}(u,v)$ be given by*

Sphere: $\mathbf{r}(u,v) = a\sin u\cos v\mathbf{i} + a\sin u\sin v\mathbf{j} + a\cos u\mathbf{k}$

[*The ranges of the parameters are* $0 \le u \le \pi$ *and* $0 \le v \le 2\pi$]

Obtain $\mathbf{r}_u, \mathbf{r}_v, \mathbf{n}, ds$.

Solution. $\mathbf{r}_u = a\cos u\cos v\mathbf{i} + a\cos u\sin v\mathbf{j} - a\sin u\mathbf{k}$

$\mathbf{r}_v = -a\sin u\sin v\mathbf{i} + a\sin u\cos v\mathbf{j}$.

$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos u\cos v & a\cos u\sin v & -a\sin u \\ -a\sin u\sin v & a\sin u\cos v & 0 \end{vmatrix} \\ &= a^2[\sin^2 u\cos v\mathbf{i} + \sin^2 u\sin v\mathbf{j} + \sin u\cos u\mathbf{k}]\end{aligned}$$

$$|\mathbf{r}_u \times \mathbf{r}_v| = a^2\sqrt{\sin^4 u\cos^2 v + \sin^4 u\sin^2 u + \sin^2 u\cos^2 u} = a^2\sin u.$$

$$\therefore \quad \mathbf{n} = \frac{\mathbf{r}_n \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} = \sin u\cos v\mathbf{i} + \sin u\sin v\mathbf{j} + \cos u\mathbf{k} = \frac{1}{a}\mathbf{r}.$$

[The positive direction of the unit normal to the sphere is outwards from the sphere.]

$$\begin{aligned} ds^2 &= Edu^2 + 2Fdu\,dv + Gdv^2 \quad (E = \mathbf{r}_u^2, G = \mathbf{r}_v^2, F = \mathbf{r}_u.\mathbf{r}_v) \\ &= a^2(du^2 + \sin^2 u dv^2) \quad \text{[check]}.\end{aligned}$$

Figure 10

4. Definition of Surface Integral–

Let $\mathbf{F}$ be a vector function of u and v defined over a region S of the surface $\mathbf{r}(u,v)$. We subdivide in any manner the surface S into u sub-regions of areas $\Delta S_1, \Delta S_2, \ldots, \Delta S_n$. We choose any point (μ_r, v_r) inside or on the boundary of the sub-region ΔS_r. We then form the sum

$$S_n = \sum_{r=1}^{1} \mathbf{F}(\mu_r, v_r).\mathbf{n}(\mu_r, v_r)\Delta S_r,$$

where $\mathbf{n}$ is the unit normal vector. Let $n \to \infty$ so that each ΔS_r shrinks to a point. If $\lim_{n\to\infty} S_n$ exists and is independent of the mode of sub-division of S and is also independent of the choice of (μ_r, ν_r) then that limit is called the **surface integral** of $\mathbf{F}(u,v)$ over the region S of the surface $\mathbf{r}(u,v)$ and is denoted by

$$\int_S \mathbf{F}.\mathbf{n}dS = \int_S \mathbf{F}.d\mathbf{S} \quad (d\mathbf{S} = \mathbf{n}dS).$$

Interpreting u, v as rectangular cartesian co-ordinates we may evalute surface integral as a double integral over a suitable region $\bar{S}$ of the uv-plane. Since $dS = |\mathbf{r}_u \times \mathbf{r}_v| du\,dv$ we may replace

$$\int_S \mathbf{F}.\mathbf{n}dS \quad \text{by}$$

$$\int_S \mathbf{F}.\mathbf{n}|\mathbf{r}_u \times \mathbf{r}_v| du\,dv.$$

Figure 11

Surface Integrals

3 types of vector surface integrals

(1) $\int_S \vec{F}\, ds = \int_S \vec{F}\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$

(2) $\int_S \vec{F} \times \hat{n}\, ds = \int_S \vec{F} \times \hat{n}\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$

(3) $\int_S f\, d\vec{s} = \int_S f\, \hat{n}\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$

$u, v \rightarrow$ parameters
$\hookrightarrow$ scalars (not vectors)

$ds = |\vec{r}_u \times \vec{r}_v|\, du\, dv$

$d\vec{s} = \hat{n}\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$

$f \rightarrow$ scalar function of u, v : $f(u, v)$.

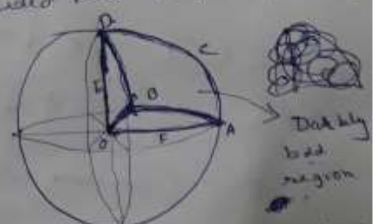
<u>Example</u> :- Let $\vec{F} = (\cos u \cos v, \cos u \sin v, -\sin u)$. Let S be the octant of the sphere $\vec{r} = (a \sin u \cos v, a \sin u \sin v, a \cos u)$ corresponding to $u \in [0, \pi/2]$ & $v \in [0, \pi/2]$. Evaluate

$\int_S \vec{F}\, ds$, $\int_S \vec{F} \cdot d\vec{s}$, $\int_S \vec{F} \times d\vec{s}$

<u>Note</u> :→ (No need to mention this for doing the sum.)

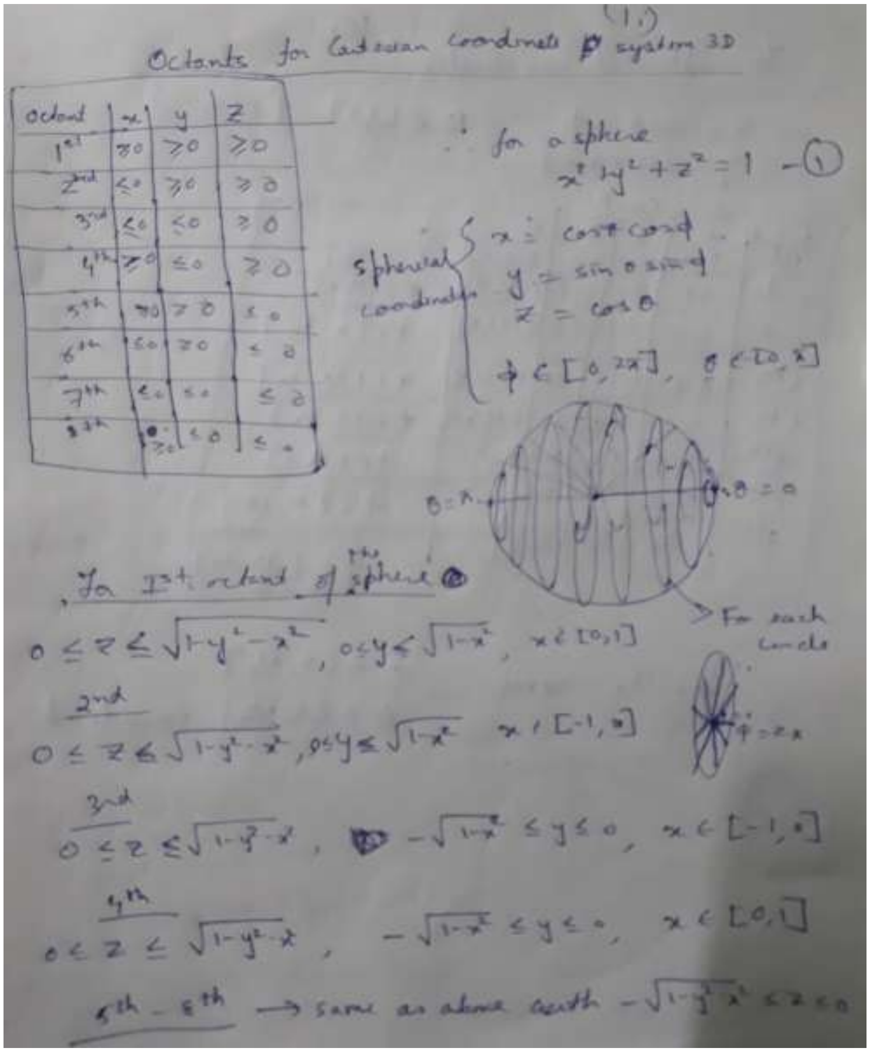
→ <u>Octant</u> → A sphere is divided into 8 equal parts by any 2 great circles

(Just like a circle is divided into 4 quadrants)



<u>Octant</u> → Any 3D coordinate system divided into 8 octants just as 2D divided into 4 quadrants

Figure 12



(1.)

Octants for Cartesian coordinate system 3D

octant	x	y	z
1st	≥0	≥0	≥0
2nd	≤0	≥0	≥0
3rd	≤0	≤0	≥0
4th	≥0	≤0	≥0
5th	≥0	≥0	≤0
6th	≤0	≥0	≤0
7th	≤0	≤0	≤0
8th	≥0	≤0	≤0

∴ for a sphere

$x^2+y^2+z^2=1$ — (1)

spherical coordinates $\left\{ \begin{array}{l} x = \cos\theta\cos\phi \\ y = \sin\theta\sin\phi \\ z = \cos\theta \\ \phi\in[0,2\pi],\ \theta\in[0,\pi] \end{array} \right.$

For 1st octant of the sphere (1)

$0\le z\le\sqrt{1-y^2-x^2}$, $0\le y\le\sqrt{1-x^2}$, $x\in[0,1]$

2nd

$0\le z\le\sqrt{1-y^2-x^2}$, $0\le y\le\sqrt{1-x^2}$ $x\in[-1,0]$

3rd

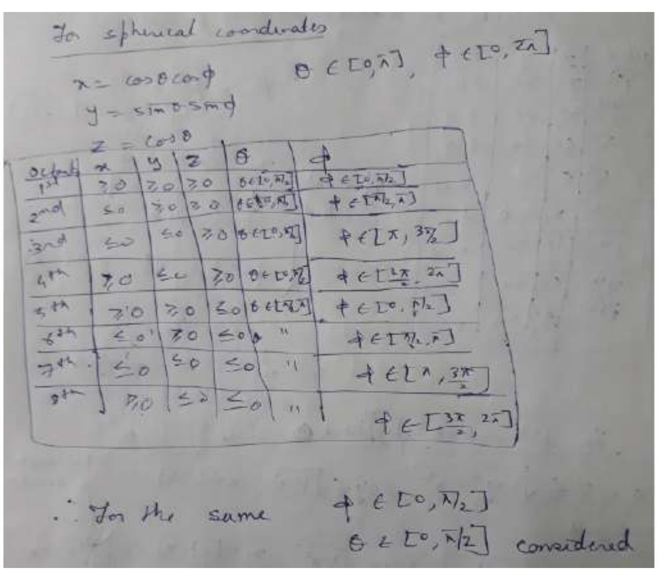
$0\le z\le\sqrt{1-y^2-x^2}$, $-\sqrt{1-x^2}\le y\le 0$, $x\in[-1,0]$

4th

$0\le z\le\sqrt{1-y^2-x^2}$, $-\sqrt{1-x^2}\le y\le 0$, $x\in[0,1]$

5th – 8th → same as above with $-\sqrt{1-y^2-x^2}\le z\le 0$

Figure 13



For spherical coordinates

$x=\cos\theta\cos\phi$ $\theta\in[0,\pi]$, $\phi\in[0,2\pi]$

$y=\sin\theta\sin\phi$

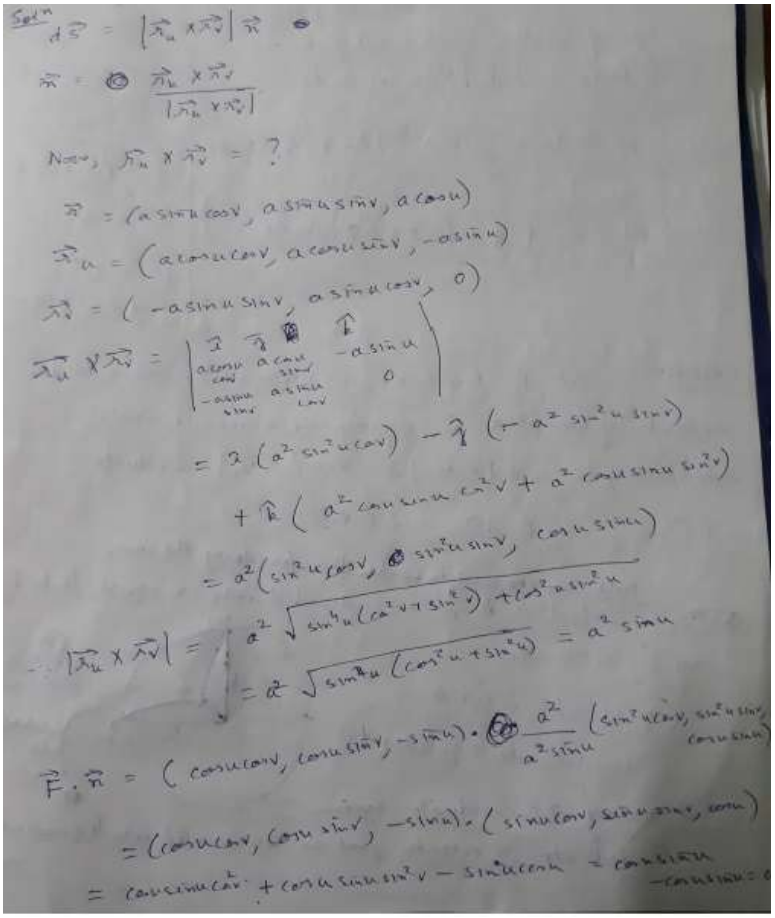
$z=\cos\theta$

octant	x	y	z	θ	φ
1st	≥0	≥0	≥0	θ∈[0,π/2]	φ∈[0,π/2]
2nd	≤0	≥0	≥0	θ∈[0,π/2]	φ∈[π/2,π]
3rd	≤0	≤0	≥0	θ∈[0,π/2]	φ∈[π,3π/2]
4th	≥0	≤0	≥0	θ∈[0,π/2]	φ∈[3π/2,2π]
5th	≥0	≥0	≤0	θ∈[π/2,π]	φ∈[0,π/2]
6th	≤0	≥0	≤0	"	φ∈[π/2,π]
7th	≤0	≤0	≤0	"	φ∈[π,3π/2]
8th	≥0	≤0	≤0	"	φ∈[3π/2,2π]

∴ For the same $\phi\in[0,\pi/2]$

$\theta\in[0,\pi/2]$ considered

Figure 14



Sol^n

$d\vec{S} = |\vec{r}_u \times \vec{r}_v|\, \hat{n}$

$\hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$

Now, $\vec{r}_u \times \vec{r}_v = ?$

$\vec{r} = (a \sin u \cos v,\ a \sin u \sin v,\ a \cos u)$

$\vec{r}_u = (a \cos u \cos v,\ a \cos u \sin v,\ -a \sin u)$

$\vec{r}_v = (-a \sin u \sin v,\ a \sin u \cos v,\ 0)$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \cos u \cos v & a \cos u \sin v & -a \sin u \\ -a \sin u \sin v & a \sin u \cos v & 0 \end{vmatrix}$$

$= \hat{i}(a^2 \sin^2 u \cos v) - \hat{j}(-a^2 \sin^2 u \sin v)$

$+ \hat{k}(a^2 \cos u \sin u \cos^2 v + a^2 \cos u \sin u \sin^2 v)$

$= a^2(\sin^2 u \cos v,\ \sin^2 u \sin v,\ \cos u \sin u)$

$\therefore |\vec{r}_u \times \vec{r}_v| = a^2 \sqrt{\sin^4 u(\cos^2 v + \sin^2 v) + \cos^2 u \sin^2 u}$

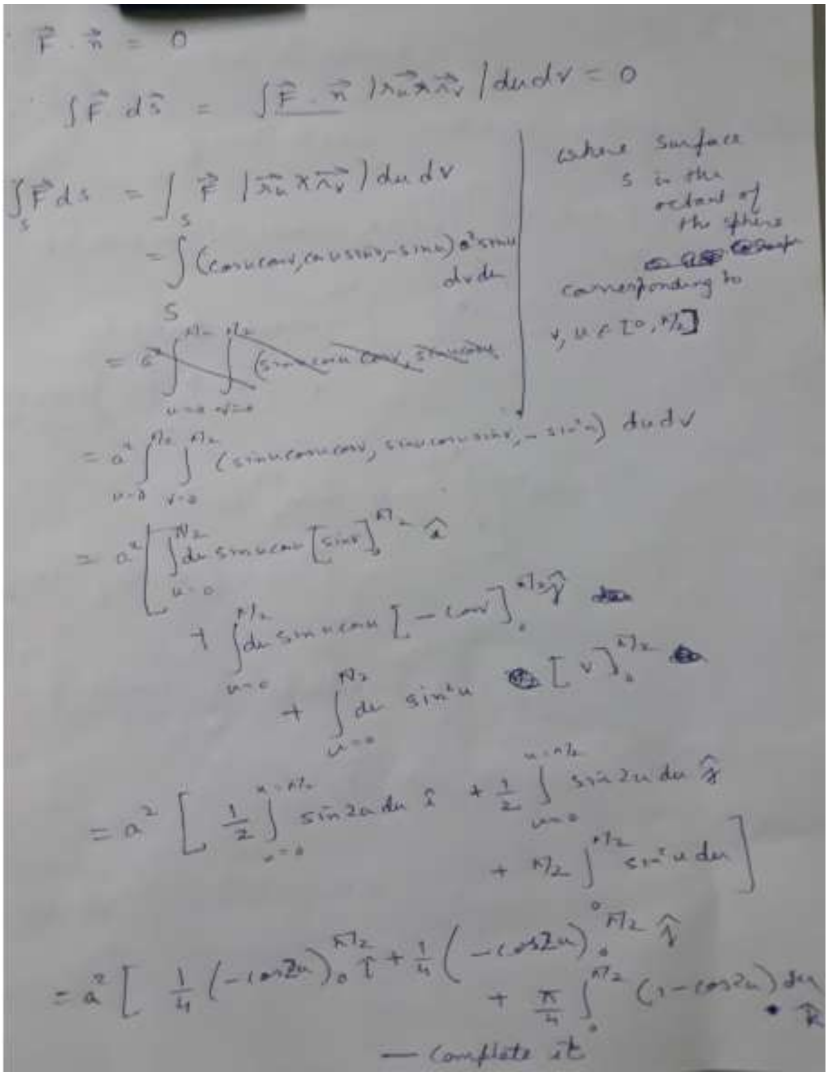
$= a^2 \sqrt{\sin^2 u(\cos^2 u + \sin^2 u)} = a^2 \sin u$

$\vec{F} \cdot \hat{n} = (\cos u \cos v,\ \cos u \sin v,\ -\sin u) \cdot \dfrac{a^2}{a^2 \sin u}(\sin^2 u \cos v,\ \sin^2 u \sin v,\ \cos u \sin u)$

$= (\cos u \cos v,\ \cos u \sin v,\ -\sin u) \cdot (\sin u \cos v,\ \sin u \sin v,\ \cos u)$

$= \cos u \sin u \cos^2 v + \cos u \sin u \sin^2 v - \sin u \cos u = \cos u \sin u - \cos u \sin u = 0$

Figure 15



$\vec{F} \cdot \vec{n} = 0$

$\int \vec{F}\, d\vec{s} = \int \vec{F} \cdot \vec{n}\, |\vec{r_u} \times \vec{r_v}|\, du\, dv = 0$

$\int_S \vec{F}\, ds = \int_S \vec{F}\, |\vec{r_u} \times \vec{r_v}|\, du\, dv$

where surface S is the octant of the sphere corresponding to $v, u \in [0, \pi/2]$

$= \int_S (\cos u \cos v, \cos u \sin v, -\sin u)\, a^2 \sin u\, dv\, du$

$= a^2 \int_{u=0}^{\pi/2} \int_{v=0}^{\pi/2} (\sin u \cos u \cos v, \sin u \cos u \sin v, -\sin^2 u)\, du\, dv$

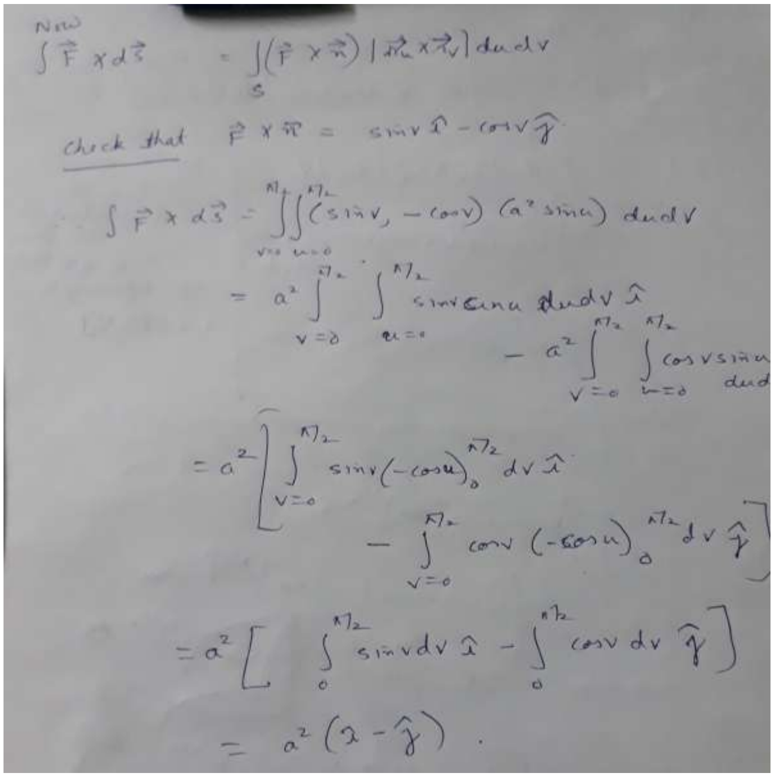
$= a^2 \Big[\int_{u=0}^{\pi/2} du \sin u \cos u\, [\sin v]_0^{\pi/2}\, \hat{x} + \int_{u=0}^{\pi/2} du \sin u \cos u\, [-\cos v]_0^{\pi/2}\, \hat{y} + \int_{u=0}^{\pi/2} du \sin^2 u\, [v]_0^{\pi/2}$

$= a^2 \Big[\frac{1}{2} \int_{u=0}^{\pi/2} \sin 2u\, du\, \hat{x} + \frac{1}{2} \int_{u=0}^{\pi/2} \sin 2u\, du\, \hat{y} + \pi/2 \int_0^{\pi/2} \sin^2 u\, du \Big]$

$= a^2 \Big[\frac{1}{4} (-\cos 2u)_0^{\pi/2}\, \hat{x} + \frac{1}{4} (-\cos 2u)_0^{\pi/2}\, \hat{y} + \frac{\pi}{4} \int_0^{\pi/2} (1 - \cos 2u)\, du\, \hat{k}$

— complete it

Figure 16



Now

$$\int \vec{F} \times d\vec{s} = \int_S (\vec{F} \times \hat{n}) \, |\vec{r}_u \times \vec{r}_v| \, du \, dv$$

Check that $\vec{F} \times \hat{n} = \sin v \, \hat{i} - \cos v \, \hat{j}$

$$\therefore \int \vec{F} \times d\vec{s} = \int_{v=0}^{\pi/2} \int_{u=0}^{\pi/2} (\sin v, -\cos v)(a^2 \sin u) \, du \, dv$$

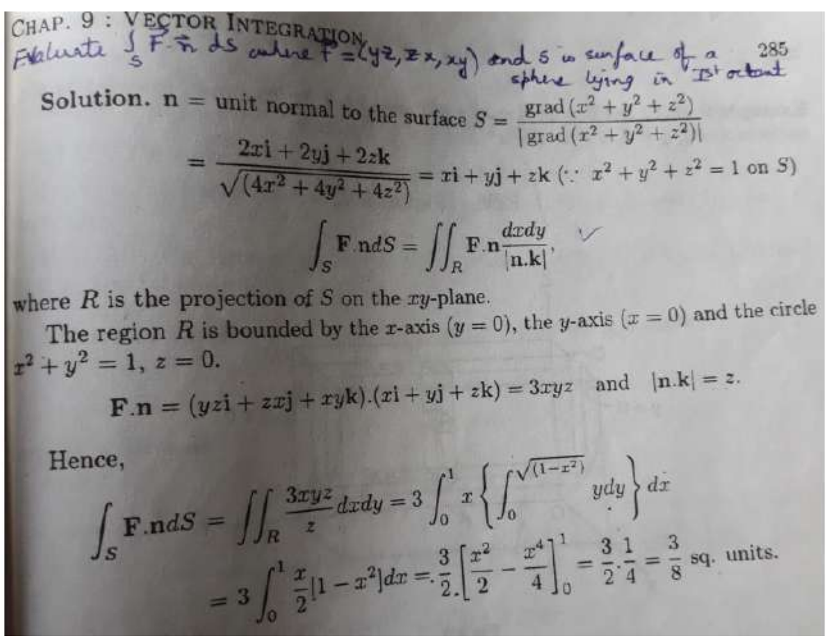
$$= a^2 \int_{v=0}^{\pi/2} \int_{u=0}^{\pi/2} \sin v \sin u \, du \, dv \, \hat{i} - a^2 \int_{v=0}^{\pi/2} \int_{u=0}^{\pi/2} \cos v \sin u \, du \, d$$

$$= a^2 \left[\int_{v=0}^{\pi/2} \sin v \, (-\cos u)_0^{\pi/2} \, dv \, \hat{i} - \int_{v=0}^{\pi/2} \cos v \, (-\cos u)_0^{\pi/2} \, dv \, \hat{j} \right]$$

$$= a^2 \left[\int_0^{\pi/2} \sin v \, dv \, \hat{i} - \int_0^{\pi/2} \cos v \, dv \, \hat{j} \right]$$

$$= a^2 (\hat{i} - \hat{j}).$$

Figure 17



CHAP. 9 : VECTOR INTEGRATION 285

Evaluate $\int_S \vec{F} \cdot \hat{n} \, ds$ where $\vec{F} = (yz, zx, xy)$ and S is surface of a sphere lying in I^{st} octant

Solution. $\mathbf{n}$ = unit normal to the surface $S = \dfrac{\text{grad}\,(x^2+y^2+z^2)}{|\text{grad}\,(x^2+y^2+z^2)|}$

$$= \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{(4x^2+4y^2+4z^2)}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad (\because x^2+y^2+z^2 = 1 \text{ on } S)$$

$$\int_S \mathbf{F}.\mathbf{n} dS = \iint_R \mathbf{F}.\mathbf{n} \frac{dx dy}{|\mathbf{n}.\mathbf{k}|},$$

where R is the projection of S on the xy-plane.

The region R is bounded by the x-axis ($y = 0$), the y-axis ($x = 0$) and the circle $x^2 + y^2 = 1$, $z = 0$.

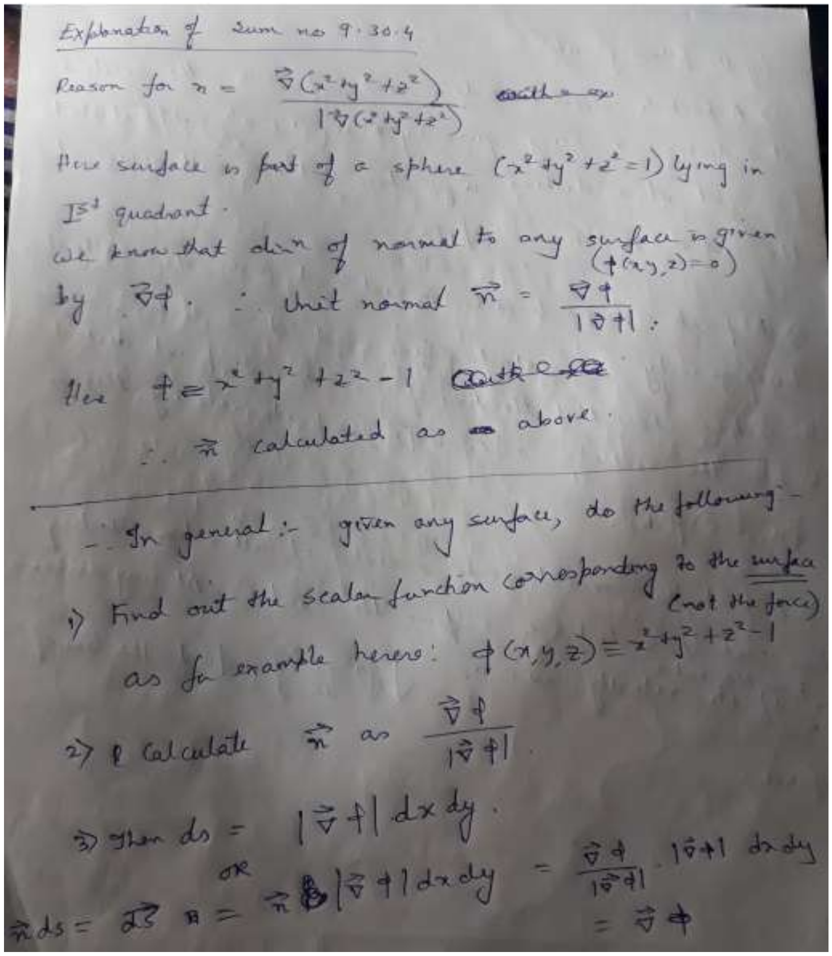
$$\mathbf{F}.\mathbf{n} = (yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}).(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 3xyz \quad \text{and} \quad |\mathbf{n}.\mathbf{k}| = z.$$

Hence,

$$\int_S \mathbf{F}.\mathbf{n} dS = \iint_R \frac{3xyz}{z} dx dy = 3\int_0^1 x \left\{ \int_0^{\sqrt{(1-x^2)}} y dy \right\} dx$$

$$= 3\int_0^1 \frac{x}{2}[1 - x^2] dx = \frac{3}{2}.\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{2}.\frac{1}{4} = \frac{3}{8} \text{ sq. units.}$$

Figure 18



Explanation of sum no 9·30·4

Reason for $\vec{n} = \dfrac{\vec{\nabla}(x^2+y^2+z^2)}{|\vec{\nabla}(x^2+y^2+z^2)|}$

Here surface is part of a sphere $(x^2+y^2+z^2=1)$ lying in I^{st} quadrant.

We know that dir^n of normal to any surface $(\phi(x,y,z)=0)$ is given by $\vec{\nabla}\phi$. $\therefore$ Unit normal $\vec{n} = \dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$.

Here $\phi = x^2+y^2+z^2-1$

$\therefore \vec{n}$ calculated as above.

$\therefore$ In general:- given any surface, do the following:-

1) Find out the scalar function corresponding to the surface (not the force)

as for example here: $\phi(x,y,z) = x^2+y^2+z^2-1$

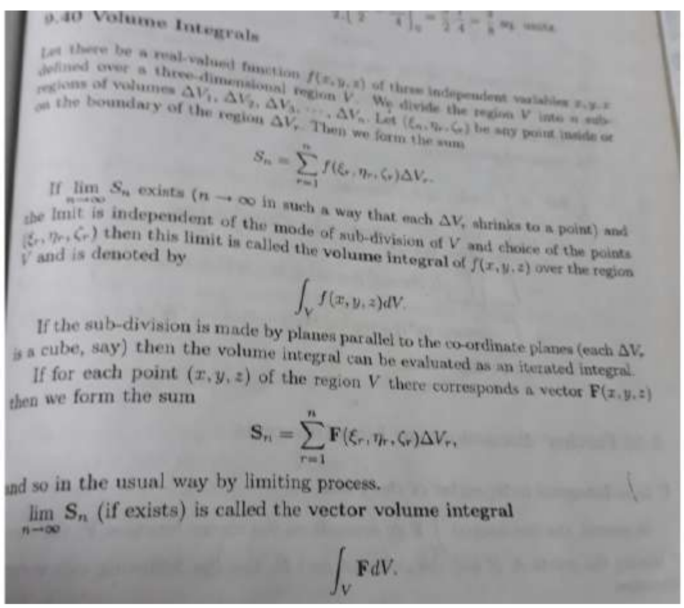
2) Calculate $\vec{n}$ as $\dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$.

3) Then $ds = |\vec{\nabla}\phi|\, dx\, dy$.

or

$\vec{n}\, ds = \vec{ds} = \vec{n}\, |\vec{\nabla}\phi|\, dx\, dy = \dfrac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} |\vec{\nabla}\phi|\, dx\, dy = \vec{\nabla}\phi$

Figure 19



9.40 Volume Integrals

Let there be a real-valued function $f(x,y,z)$ of three independent variables x,y,z defined over a three-dimensional region V. We divide the region V into n sub-regions of volumes $\Delta V_1, \Delta V_2, \Delta V_3, \cdots, \Delta V_n$. Let (ξ_r, η_r, ζ_r) be any point inside or on the boundary of the region ΔV_r. Then we form the sum

$$S_n = \sum_{r=1}^{n} f(\xi_r, \eta_r, \zeta_r)\Delta V_r.$$

If $\lim_{n\to\infty} S_n$ exists ($n \to \infty$ in such a way that each ΔV_r shrinks to a point) and the limit is independent of the mode of sub-division of V and choice of the points (ξ_r, η_r, ζ_r) then this limit is called the **volume integral** of $f(x,y,z)$ over the region V and is denoted by

$$\int_V f(x,y,z)dV.$$

If the sub-division is made by planes parallel to the co-ordinate planes (each ΔV_r is a cube, say) then the volume integral can be evaluated as an iterated integral.

If for each point (x,y,z) of the region V there corresponds a vector $\mathbf{F}(x,y,z)$ then we form the sum

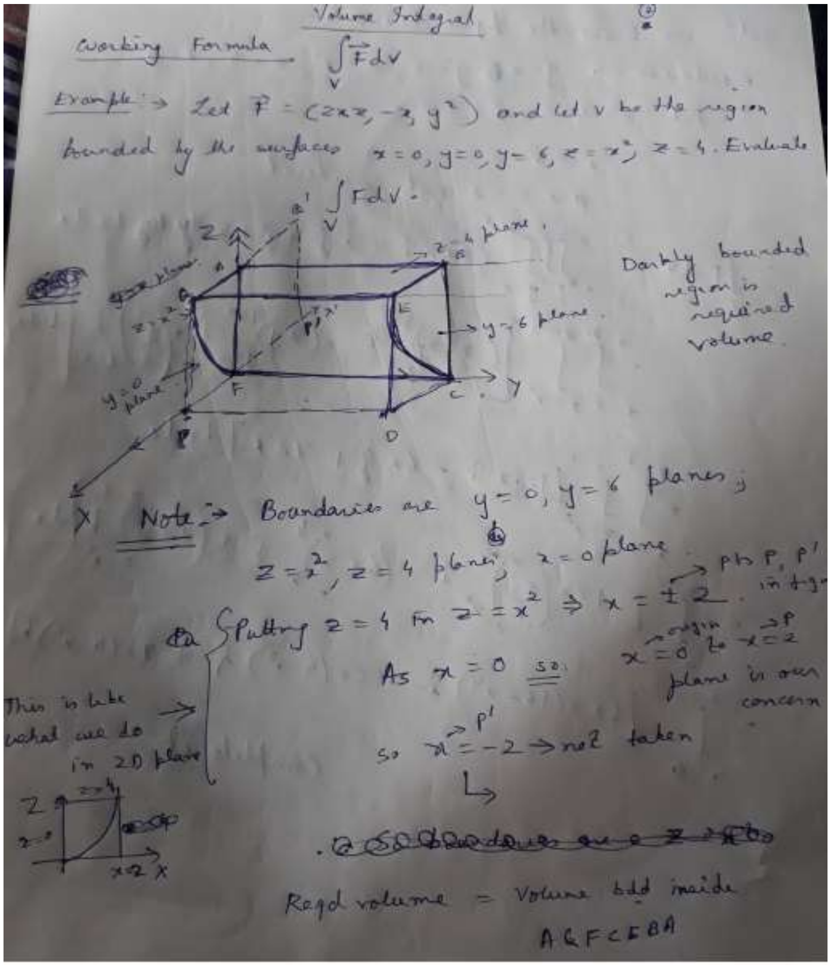
$$\mathbf{S}_n = \sum_{r=1}^{n} \mathbf{F}(\xi_r, \eta_r, \zeta_r)\Delta V_r,$$

and so in the usual way by limiting process.

$\lim_{n\to\infty} \mathbf{S}_n$ (if exists) is called the **vector volume integral**

$$\int_V \mathbf{F}dV.$$

Figure 20



Volume Integral

Working Formula $\int_V \vec{F}\,dV$

Example → Let $\vec{F} = (2xz, -x, y^2)$ and let V be the region bounded by the surfaces $x=0, y=0, y=6, z=x^2, z=4$. Evaluate $\int_V F\,dV$.

Darkly bounded region is required volume

Note → Boundaries are $y=0$, $y=6$ planes; $z=x^2$, $z=4$ planes, $x=0$ plane

Putting $z=4$ in $z=x^2 \Rightarrow x = \pm 2$ → pts P, P' in +y

As $x=0$ so origin $x=0$ to $x=2$ → P plane is our concern

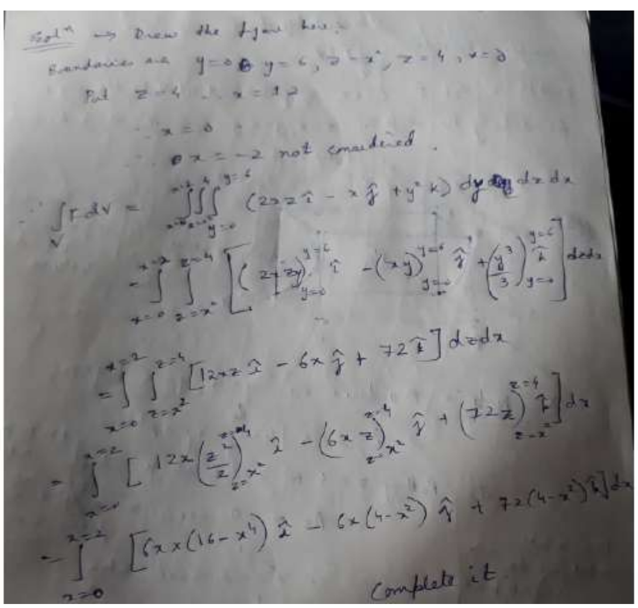
So $x=-2$ → P' → not taken

This is like what we do in 2D plane

Z, $z=4$, $z=0$, $x=2$, X

Reqd volume = Volume bdd inside AGFCEBA

Figure 21



Solⁿ → Draw the figure here.

Boundaries are $y=0$, $y=6$, $z=x^2$, $z=4$, $x=0$

Put $z=4$ ∴ $x=\pm 2$

∴ $x=0$

$x=-2$ not considered.

$$\int_V F\,dV = \int_{x=0}^{x=2}\int_{z=x^2}^{4}\int_{y=0}^{y=6} (2xz\,\hat{i} - x\,\hat{j} + y^2\,\hat{k})\,dy\,dz\,dx$$

$$= \int_{x=0}^{x=2}\int_{z=x^2}^{z=4}\left[(2xzy)_{y=0}^{y=6}\hat{i} - (xy)_{y=0}^{y=6}\hat{j} + \left(\frac{y^3}{3}\right)_{y=0}^{y=6}\hat{k}\right]dz\,dx$$

$$= \int_{x=0}^{x=2}\int_{z=x^2}^{z=4}\left[12xz\,\hat{i} - 6x\,\hat{j} + 72\,\hat{k}\right]dz\,dx$$

$$= \int_{x=0}^{x=2}\left[12x\left(\frac{z^2}{2}\right)_{z=x^2}^{z=4}\hat{i} - (6xz)_{z=x^2}^{z=4}\hat{j} + (72z)_{z=x^2}^{z=4}\hat{k}\right]dx$$

$$= \int_{x=0}^{x=2}\left[6x(16-x^4)\,\hat{i} - 6x(4-x^2)\,\hat{j} + 72(4-x^2)\,\hat{k}\right]dx$$

Complete it

Figure 22