Plane Electromagnetic waves in a conducting Medium

Let us consider linear, homogeneous & isotropic conducting medium where

ϵ = permittivity.
μ = permeability.
σ = conductivity.
$\rho = 0 \Rightarrow$ for free charge density when ~~supp~~ ext. applied field = 0.

Maxwell's eqns. $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon = 0$.

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}.$$

We know, from (15) & (16).

$$\vec{\nabla}^2 \vec{H} = \sigma \mu \frac{\partial \vec{H}}{\partial t} + \mu \epsilon \frac{\partial^2 \vec{H}}{\partial t^2}$$

$$\& \ \nabla^2 \vec{E} = \sigma \mu \frac{\partial \vec{E}}{\partial t} + \mu \epsilon \frac{\partial^2 \vec{H}}{\partial t^2}$$

Since EM wave is sinusoidal, so its solution will be as in the form.

$$\vec{E}(\vec{r}, t) = \vec{E_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{---(20)}$$

$$\vec{H}(\vec{r}, t) = \vec{H_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{---(21)}.$$

where $\vec{E_0}$ = complex amplitude of electric field
$\vec{H_0}$ = " " " magnetic ".
$\vec{r}$ = position vector
t = time.
$\vec{k}$ = propagation vector
ω = angular freq.

Considering $\vec{E}(\vec{r}, t) = \vec{E_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)}$

$$\frac{\partial \vec{E}}{\partial t} = \vec{E_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)} (-j\omega).$$
$$= -j\omega \vec{E_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = -j\omega \vec{E_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)} \cdot (-j\omega)$$
$$= +\omega^2 \vec{E_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)} (-1)$$

$$\boxed{\frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)}}$$

$$\vec{\nabla}\cdot\vec{E} = \vec{E_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)} \cdot (jk).$$

$$\nabla^2 \vec{E} = jk\vec{E_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)} (jk)$$
$$= -k^2 \vec{E_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)}.$$

putting these in eqn (16) $\nabla^2 \vec{E} = \sigma\mu \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$.

or, $-k^2 \vec{E_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)} = \sigma\mu\left[-j\omega \vec{E_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)}\right] + \mu\epsilon\left[-\omega^2 \vec{E_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)}\right]$

$$= (-j\sigma\mu\omega - \mu\epsilon\omega^2)\, \vec{E_0}\, e^{j(\vec{k}\cdot\vec{r} - \omega t)}$$

or, $-k^2 \vec{E} = (-j\sigma\mu\omega - \mu\epsilon\omega^2)\, \vec{E}$

or, $\left[-k^2 + j\sigma\mu\omega + \mu\epsilon\omega^2\right] \vec{E} = 0$.

$\because \vec{E} \neq 0$, then $-k^2 + j\sigma\mu\omega + \mu\epsilon\omega^2 = 0$.

or, $\boxed{k^2 = j\sigma\mu\omega + \mu\epsilon\omega^2}$ — (22).

From eqn (22), it is clear that the term 'k' has two parts $\rightarrow$ real part $= \epsilon\mu\omega^2$
$\rightarrow$ imaginary part $= j\sigma\mu\omega$.

So k $\rightarrow$ is a complex character.

Let, $k = \alpha + j\beta$, where $\boxed{\begin{aligned}\alpha &= \epsilon\mu\omega^2 \\ \beta &= \sigma\mu\omega\end{aligned}}$ — (23).

then, $k^2 = (\alpha + j\beta)^2$

$= \alpha^2 + (j\beta)^2 + 2\alpha j\beta$.

$k^2 = \alpha^2 - \beta^2 + 2j\alpha\beta$. — (24).

Comparing (22) & (24).

$(\alpha^2 - \beta^2) = \epsilon\mu\omega^2$ & $2\alpha\beta = \sigma\mu\omega$ — (25B)

(25A)

or, $\beta = \frac{\sigma\mu\omega}{2\alpha}$.

$$(\alpha^2+\beta^2)^2 = (\alpha^2-\beta^2)^2 + 4\alpha^2\beta^2$$

$$= \epsilon\mu\omega^2 + \sigma^2\mu^2\omega^2$$

$$\text{or } (\alpha^2+\beta^2) = \sqrt{\epsilon\mu\omega^2 + \sigma^2\mu^2\omega^2} \quad -(26)$$

$$(25A)+(26) \Rightarrow \alpha^2-\beta^2 = \epsilon\mu\omega^2$$

$$\alpha^2+\beta^2 = \sqrt{\epsilon\mu\omega^2 + \sigma^2\mu^2\omega^2}$$

$$2\alpha^2 = \epsilon\mu\omega^2 + \sqrt{\sigma^2\mu^2\omega^2 + \epsilon\mu\omega^2}$$

$$\boxed{\text{or, } 4\beta^4 + 4\epsilon\mu\omega^2\beta^2 - \sigma^2\mu^2\omega^2 = 0.}$$

$$\alpha^2 = \frac{\epsilon\mu\omega^2 + \sqrt{\sigma^2\mu^2\omega^2 + \epsilon\mu\omega^2}}{2}$$

$$\text{or, } \alpha = \omega\sqrt{\frac{\epsilon\mu}{2}\left[1+\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}\right.}$$

$$\alpha = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}+1\right]^{1/2} \quad \text{---(27)}$$

$$\beta = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]^{1/2} \quad \text{---(28)}$$

The ratio $\frac{\sigma}{\epsilon\omega}$ is significant in eqn (27) & (28). Because $1/\omega$ has the dimension of time.

$$\therefore \tau = \frac{\epsilon}{\sigma} = \frac{\text{permittivity of the medium}}{\text{conductivity " " "}} \quad \text{---(29)}$$

This quantity is a ratio of two time scales.

Now,
$$\vec{E}(\vec{r},t) = \vec{E_0}\, e^{j(\vec{k}\cdot\vec{r}-\omega t)}$$
$$= \vec{E_0}\, e^{j[(\alpha+j\beta)\hat{n}\cdot\vec{r}-\omega t]}$$
$$= \vec{E_0}\, e^{j[(\alpha\hat{n}\cdot\vec{r}+j\beta\hat{n}\cdot\vec{r})-\omega t]}$$
$$= \vec{E_0}\, e^{[j\alpha\hat{n}\cdot\vec{r}+j^2\beta\hat{n}\cdot\vec{r}-j\omega t]}$$
$$= \vec{E_0}\, e^{[j\alpha\hat{n}\cdot\vec{r}-\beta\hat{n}\cdot\vec{r}-j\omega t]}, \quad \because j^2=-1$$
$$= \vec{E_0}\, e^{-\beta\hat{n}\cdot\vec{r}}\, e^{j(\alpha\hat{n}\cdot\vec{r}-\omega t)} \quad \text{---(30)}$$

Simillarly,
$$\vec{H}(\vec{r},t) = \vec{H_0}\, e^{-\beta\hat{n}\cdot\vec{r}}\, e^{j(\alpha\hat{n}\cdot\vec{r}-\omega t)} \quad \text{---(31)}$$

From eqn (30) & (31) $\Rightarrow$ it is clear that EM wave propagates, but with decreasing amplitude with the term $e^{-\beta\hat{n}\cdot\vec{r}}$

The quantity β, the imaginary part of the wave number k is the measure of decrement and is called attenuation constant.

$\beta = \sigma\mu\omega \rightarrow \beta$ depends on conductivity σ, permeability μ, ang. frequency ω.

The physical reason of this attenuation is that the wave produces electric current in the medium, which causes dissipation of energy in the form of Joule heating.

For good conductor, $\sigma \gg 1$, $\therefore \frac{\sigma}{\omega\epsilon} \gg 1$.

Using this condition in eqn (28).

$$\beta = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2}-1\right]^{1/2}$$

$$= \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{\left(\frac{\sigma}{\omega\epsilon}\right)^2}\right]^{1/2}$$

$$= \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\frac{\sigma}{\omega\epsilon}}. \quad \left[\because \left[\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/2} = \left(\frac{\sigma}{\omega\epsilon}\right)^1\right].$$

$$= \cancel{\omega}^{\,\omega^{1/2}} \times \frac{\cancel{\epsilon^{1/2}}\mu^{1/2}}{2^{1/2}} \times \frac{\sigma^{1/2}}{\cancel{\omega^{1/2}}\cancel{\epsilon^{1/2}}}$$

$$= \frac{\omega^{1/2}\mu^{1/2}\sigma^{1/2}}{2^{1/2}} = \sqrt{\frac{\omega\sigma\mu}{2}}.$$

$$\therefore \boxed{\beta = \sqrt{\frac{\omega\sigma\mu}{2}}}$$

Similarly. we get $\boxed{\alpha = \sqrt{\frac{\omega\sigma\mu}{2}}}$

It proves that for good conductor, $\boxed{\alpha = \beta = \sqrt{\frac{\omega\sigma\mu}{2}}}$

The quantity $1/\beta$ measures the depth at which electromagnetic wave entering a conductor is attenuated to $1/e$ of its initial amplitude at the surface. — it is known as skin depth or penetration depth into a conducting medium.

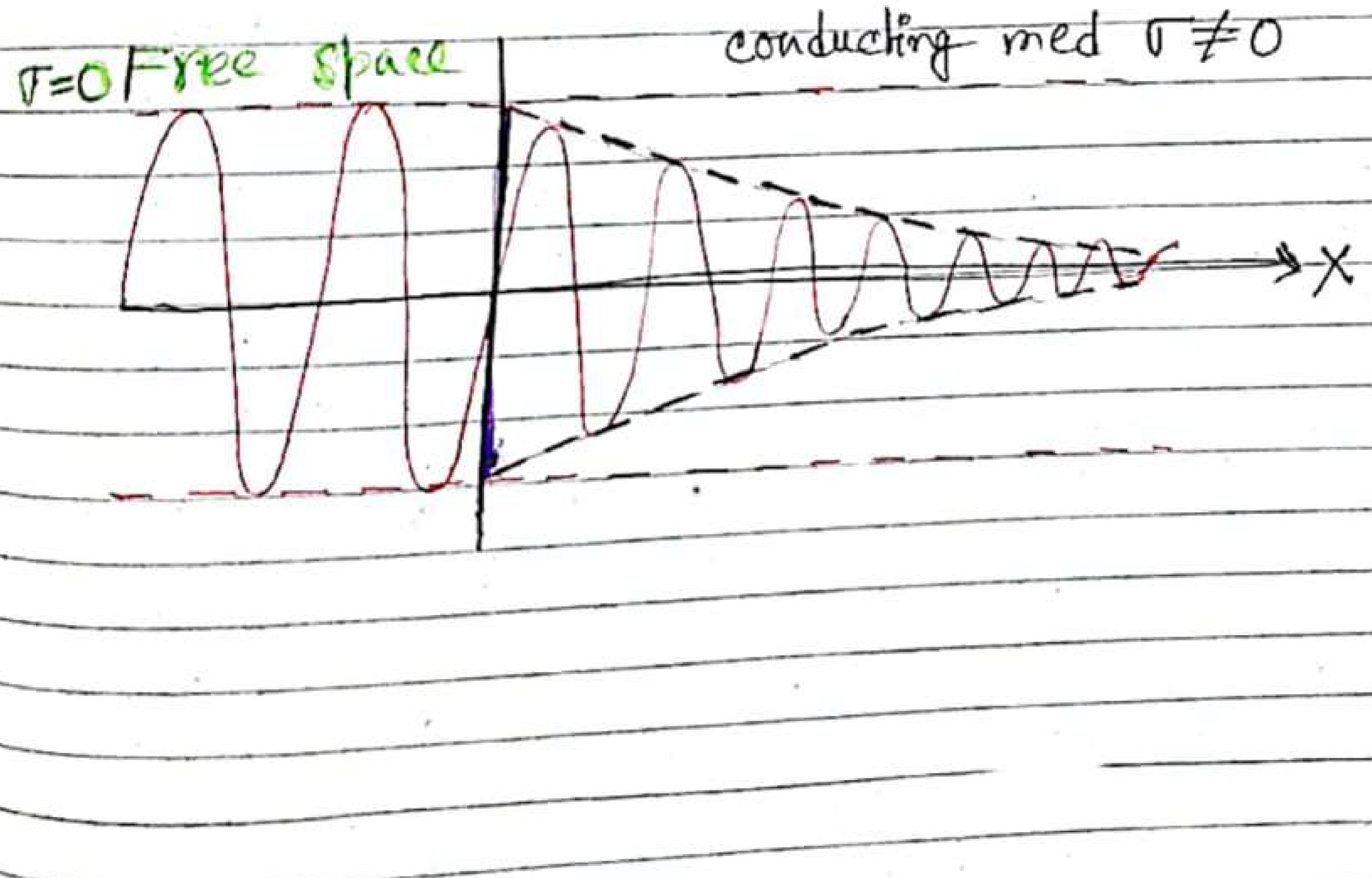
skin depth $= \delta = 1/\beta = \sqrt{\frac{2}{\omega\sigma\mu}}$.

Thus $\delta \propto \frac{1}{\sqrt{\omega}}$

$\propto \frac{1}{\sqrt{\sigma}}$

$\propto \frac{1}{\sqrt{\mu}}$.

Due to this, ac resistance of a conductor > dc.



tion of the field vectors.

For goods conductors $\sigma/\omega\epsilon \gg 1$ and then we have

$$\alpha \simeq \beta \approx \sqrt{\frac{\omega\sigma\mu}{2}}. \tag{15.7-12}$$

The quantity $1/\beta$ measures the depth at which electromagnetic wave entering a conductor is attenuated to $1/e$ of its initial amplitude at the surface. It is known as *skin depth* or *penetration depth* into a conducting medium. Thus, skin depth δ for a good conductor is

Free space

conducting med

$$\boxed{\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\omega\sigma\mu}}.} \tag{15.7-13}$$

Amplitude of EM wave decreases exponentially when it travels through conducting med

Obviously, skin depth δ decreases with increase in frequency and conductivity. For good conductors at high frequencies δ is very small. That is why in high frequency circuits current flows only through the surface of good conductors. This phenomenon is called *skin effect*. Due to this effect the ac resistance of a conductor is greater than its dc resistance. For this in high frequency circuits it is better to use a number of fine stranded wires instead of a thick wire. It increases surface area for a given area of cross-section and reduces resistance. At microwave frequencies δ for Ag is very small ($\sim 10^{-3}$ mm). As a result, in the microwave region the performance of a waveguide made of pure Ag and another waveguide made of Ag-coated brass would appear to be identical. This technique is used to reduce the material cost of good conductors.

The attenuation of electromagnetic waves in the conducting sea water creates problem in radiocommunication with submerged submarine.

Wavelength, propagation speed and the index of refraction

The real part of k, i.e., α determines the wavelength, the propagation speed of the wave and the index of refraction of the conductor, in the usual way. Thus,

$$\lambda_c = \frac{2\pi}{\alpha}, \quad v = \frac{\omega}{\alpha} \quad \text{and} \quad \boxed{n = \frac{c\alpha}{\omega}}$$

For a good conductor $\alpha = \sqrt{\frac{\omega\sigma\mu}{2}}$ and hence,

$$\lambda_c = 2\pi\sqrt{\frac{2}{\omega\sigma\mu}}, \quad v = \omega\sqrt{\frac{2}{\omega\sigma\mu}} = \sqrt{\frac{2\omega}{\sigma\mu}} \quad \text{and} \quad n = c\sqrt{\frac{\sigma\mu}{2\omega}}.$$

Note that the skin depth δ as given by (15.7-13) for a good conductor can be expressed in terms of the wavelength λ_c in the conductor as

$$\boxed{\delta = \frac{\lambda_c}{2\pi}.}$$

Relative directions of $\vec{E}$, $\vec{H}$ and $\vec{k}$

Substituting the solutions (15.7-7) in Eqs. (15.7-1) and (15.7-2) we get

$$\vec{k}\cdot\vec{E} = 0 \quad \text{and} \quad \vec{k}\cdot\vec{H} = 0. \tag{15.7-14}$$

These equations indicates that $\vec{E}$ and $\vec{H}$ are both perpendicular to the direction of propagation. So *electromagnetic waves in a conducting medium are transverse in nature.*

Again, the substitution of the solutions (15.7-7) in Eqs. (15.7-3) and (15.7-4) gives

$$j\vec{k}\times\vec{E} = -\mu\left(-j\omega\vec{H}\right) \quad \text{or} \quad \vec{k}\times\vec{E} = \mu\omega\vec{H} \tag{15.7-15}$$

$$\text{and} \quad j\vec{k}\times\vec{H} = \sigma\vec{E} - j\omega\epsilon\vec{E} \quad \text{or} \quad \vec{k}\times\vec{H} = -\left(\omega\epsilon + j\sigma\right)\vec{E}. \tag{15.7-16}$$

These two equations imply that $\vec{E}$ and $\vec{H}$ are mutually perpendicular and also they are perpendicular to the direction of propagation vector $\vec{k}$.

Relative phase of $\vec{E}$ and $\vec{H}$

From Eq. (15.7-15) we have

$$\vec{H} = \frac{1}{\mu\omega}\left(\vec{k}\times\vec{E}\right) = \frac{k}{\mu\omega}\left(\hat{n}\times\vec{E}\right) = \frac{\alpha + j\beta}{\mu\omega}\left(\hat{n}\times\vec{E}\right). \tag{15.7-17}$$

This equation shows that *$\vec{E}$ and $\vec{H}$ are not in phase in a conductor.*

Writing $\alpha + j\beta = \sqrt{\alpha^2+\beta^2}e^{j\phi}$; $\phi = \tan^{-1}(\beta/\alpha)$ and using (15.7-7), Eq. (15.7-17) may be rewritten as

$$\vec{H} = \frac{\sqrt{\alpha^2+\beta^2}}{\mu\omega}\left(\hat{n}\times\vec{E}_0\right)\cdot e^{j\left(\vec{k}\cdot\vec{r}-\omega t-\phi\right)}, \tag{15.7-18}$$

where

$$\sqrt{\alpha^2+\beta^2} = \omega\sqrt{\epsilon\mu}\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4}.$$

Thus, *$\vec{H}$ lags behind $\vec{E}$ in time by the phase angle*

$$\phi = \tan^{-1}\frac{\beta}{\alpha} = \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\epsilon}\right).$$

For good conductors $\alpha \approx \beta$ and $\phi = 45°$. Therefore, *the phase difference between the $\vec{E}$ and $\vec{H}$ fields in a perfect conductor is* $45°$.

Relative magnitudes of $\vec{E}$ and $\vec{H}$ is

$$\frac{\left|\vec{H}\right|}{\left|\vec{E}\right|} = \frac{H_0}{E_0} = \frac{\sqrt{\alpha^2+\beta^2}}{\mu\omega} = \sqrt{\frac{\epsilon}{\mu}}\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4}. \quad (15.7\text{-}19)$$

For good conductors

$$\frac{\left|\vec{H}\right|}{\left|\vec{E}\right|} = \sqrt{\frac{\sigma}{\omega\mu}}.$$

Thus, in this case,

$$\left|\vec{H}\right| \gg \left|\vec{E}\right|,$$

which indicates that in *a good conducting medium the field energy is not equally shared between $\vec{E}$- and $\vec{H}$-fields but it is almost entirely magnetic in nature.*

Poynting's vector

The time average Poynting's vector is

$$\langle\vec{s}\rangle = \tfrac{1}{2}Re\left(\vec{E}\times\vec{H}^*\right).$$

Using Eq. (15.7-17),

$$\begin{aligned}\langle\vec{s}\rangle &= \tfrac{1}{2}\frac{\sqrt{\alpha^2+\beta^2}}{\mu\omega}\cdot Re\left[\vec{E}\times\left(\hat{n}\times\vec{E}^*\right)\right]e^{-j\phi}\\ &= \frac{\sqrt{\alpha^2+\beta^2}}{2\mu\omega}Re\left[\hat{n}\left(\vec{E}\cdot\vec{E}^*\right)-\vec{E}^*\left(\vec{E}\cdot\hat{n}\right)\right]e^{-j\phi}\end{aligned}$$

Now putting $\vec{E}\cdot\hat{n}=0$ and using Eq. (15.7-11) we get

$$\langle\vec{s}\rangle = \frac{\sqrt{\alpha^2+\beta^2}}{2\mu\omega}\cdot E_0^2 e^{-2\beta\hat{n}\cdot\vec{r}}\cdot\cos\phi\cdot\hat{n}. \quad (15.7\text{-}20)$$

For good conductors $\alpha \simeq \beta \approx \sqrt{\dfrac{\omega\sigma\mu}{2}}$ and $\phi = 45°$. Then

$$\langle\vec{s}\rangle = \frac{1}{2}\cdot\sqrt{\frac{\sigma}{2\mu\omega}}E_0^2 e^{-2\beta\hat{n}\cdot\vec{r}}\hat{n}. \quad (15.7\text{-}21)$$

Thus, *energy flow is along the direction of propagation of the wave and is damped off exponentially.*

Average electric energy density is

$$\langle u_e \rangle = \tfrac{1}{2} Re \tfrac{1}{2} \left(\vec{E} \cdot \vec{D}^* \right) = \tfrac{1}{4} \epsilon Re \left(\vec{E} \cdot \vec{E}^* \right) = \tfrac{1}{4} \epsilon E_0^2 \cdot e^{-2\beta \hat{n} \cdot \vec{r}}.$$

Average magnetic energy density is

$$\langle u_m \rangle = \tfrac{1}{2} Re \tfrac{1}{2} \left(\vec{B} \cdot \vec{H}^* \right) = \tfrac{1}{4} \mu Re \left(\vec{H} \cdot \vec{H}^* \right) = \tfrac{1}{4} \mu H_0^2 e^{-2\beta \hat{n} \cdot \vec{r}}.$$

Thus,

$$\frac{\langle u_m \rangle}{\langle u_e \rangle} = \frac{\mu H_0^2}{\epsilon E_0^2} = \left[1 + \left(\frac{\sigma}{\omega \epsilon} \right)^2 \right]^{1/2} \tag{15.7-22}$$

Obviously, for good conductors $\langle u_m \rangle \gg \langle u_e \rangle$, i.e., *the field energy inside is almost entirely magnetic in nature.*

Total time averaged energy density is

$$\langle u \rangle = \langle u_e \rangle + \langle u_m \rangle = \tfrac{1}{4} e^{-2\beta \hat{n} \cdot \vec{r}} \left[\epsilon E_0^2 + \mu H_0^2 \right]$$

Using Eq. (15.7-19) we can write

$$\langle u \rangle = \tfrac{1}{4} e^{-2\beta \hat{n} \cdot \vec{r}} \cdot \epsilon E_0^2 \left[1 + \sqrt{1 + \left(\frac{\sigma}{\epsilon \omega} \right)^2} \right]. \tag{15.7-23}$$

In term of α, the real part of wave number as given by (15.7-10), it can be expressed as

$$\langle u \rangle = \frac{\alpha^2}{2\mu \omega^2} E_0^2 e^{-2\beta \hat{n} \cdot \vec{r}}. \tag{15.7-24}$$