(6)

EM Wave Equation from Maxwell's Equations:-

Maxwell's Equations:-

(1) $\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}$ → Gauss law of Electricity.

(2) $\vec{\nabla}.\vec{B} = 0$ → Gauss's law of magnetism

(3) $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ → Faraday's law.

(4) $\vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}$ → Maxwell's modification of Ampere's circuital Law.

From eqn (3) ⇒ variable magnetic field $\vec{B}$ produce $\vec{E}$.

From eqn (4) ⇒ " electric " $\vec{E}$ " $\vec{B}$.

Maxwell Concluded that → (a) $\vec{E}$ & $\vec{B}$ mutually $\perp^r$ to each other i.e. $\vec{E}\perp\vec{B}$

(b) $\vec{E}$ & $\vec{B}$ propagate like wave.

(c) So $\vec{E}$ & $\vec{B}$ have amplitude, freq. & time period.

we know, wave eqn $\frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$

Considering 3-D, wave eqn in general,

$$\boxed{\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}} \quad \text{—(13)}.$$

Maxwell's eqn.

$$\vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}.$$

or, $\vec{\nabla}\times\vec{B} = \mu\left(\vec{J} + \epsilon\frac{\partial\vec{E}}{\partial t}\right)$.

or, $\vec{\nabla}\times\frac{\vec{B}}{\mu} = \vec{J} + \frac{\partial(\epsilon\vec{E})}{\partial t}$.

or, $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$ $[\because \vec{B} = \mu\vec{H}$ & $\vec{D} = \epsilon\vec{E}]$.

Taking $\vec{\nabla}\times$ on both sides,

$$\vec{\nabla}\times\vec{\nabla}\times\vec{H} = \vec{\nabla}\times\left(\vec{J}+\frac{\partial \vec{D}}{\partial t}\right)$$

$$= \vec{\nabla}\times\vec{J} + \vec{\nabla}\times\frac{\partial \vec{D}}{\partial t}$$

$$= \vec{\nabla}\times(\sigma\vec{E}) + \vec{\nabla}\times\frac{\partial(\epsilon\vec{E})}{\partial t} \quad \left[\because \vec{J}=\sigma\vec{E},\ \vec{D}=\epsilon\vec{E}\right]$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \sigma(\vec{\nabla}\times\vec{E}) + \epsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

or, $$\vec{\nabla}(\vec{\nabla}\cdot\vec{H}) - \nabla^2\vec{H} = \sigma(\vec{\nabla}\times\vec{E}) + \epsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$= \sigma\left(-\frac{\partial \vec{B}}{\partial t}\right) + \epsilon\frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\boxed{\vec{\nabla}(\vec{\nabla}\cdot\vec{H}) - \nabla^2\vec{H} = -\sigma\frac{\partial \vec{B}}{\partial t} - \epsilon\frac{\partial^2 \vec{B}}{\partial t^2}} \quad -(14)$$

Now, $\vec{\nabla}\cdot\vec{H} = \vec{\nabla}\cdot\left(\frac{\vec{B}}{\mu}\right) = \frac{1}{\mu}(\vec{\nabla}\cdot\vec{B}) = \frac{1}{\mu}\times 0 \quad [\because \vec{\nabla}\cdot\vec{B}=0]$

$\therefore \boxed{\vec{\nabla}\cdot\vec{H} = 0}$

$\therefore$ eqn (14) becomes $\Rightarrow$:-

$$-\nabla^2\vec{H} = -\sigma\frac{\partial \vec{B}}{\partial t} - \epsilon\frac{\partial^2 \vec{B}}{\partial t^2}$$

or, $$\cancel{-}\nabla^2\vec{H} = \cancel{-}\sigma\mu\frac{\partial \vec{H}}{\partial t} \cancel{-}\mu\epsilon\frac{\partial^2 \vec{H}}{\partial t^2} \quad [\because \vec{B}=\mu\vec{H}]$$

or, $$\boxed{\nabla^2\vec{H} = \sigma\mu\frac{\partial \vec{H}}{\partial t} + \mu\epsilon\frac{\partial^2 \vec{H}}{\partial t^2}} \quad -(15)$$

eqn (15) is the EM wave eqn for $\vec{H}$.

Similarly; we get, $\boxed{\nabla^2\vec{E} = \sigma\mu\frac{\partial \vec{E}}{\partial t} + \mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2}}$ $-(16)$

eqn (16) is the EM wave eqn for $\vec{E}$.

For Non-conducting medium, $\sigma = 0$,

then eqn (15) $\rightarrow \nabla^2\vec{H} = \mu\epsilon\frac{\partial^2 \vec{H}}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 \vec{H}}{\partial t^2}$ $-(17)$

& eqn (16) $\rightarrow \nabla^2\vec{E} = \mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 \vec{E}}{\partial t^2}$ $-(18)$

where $\mu\epsilon = \frac{1}{v^2}$

or, $\boxed{v = \frac{1}{\sqrt{\mu\epsilon}}}$ $-(19)$

eqn (17) & (18) are similar to eqn (13)