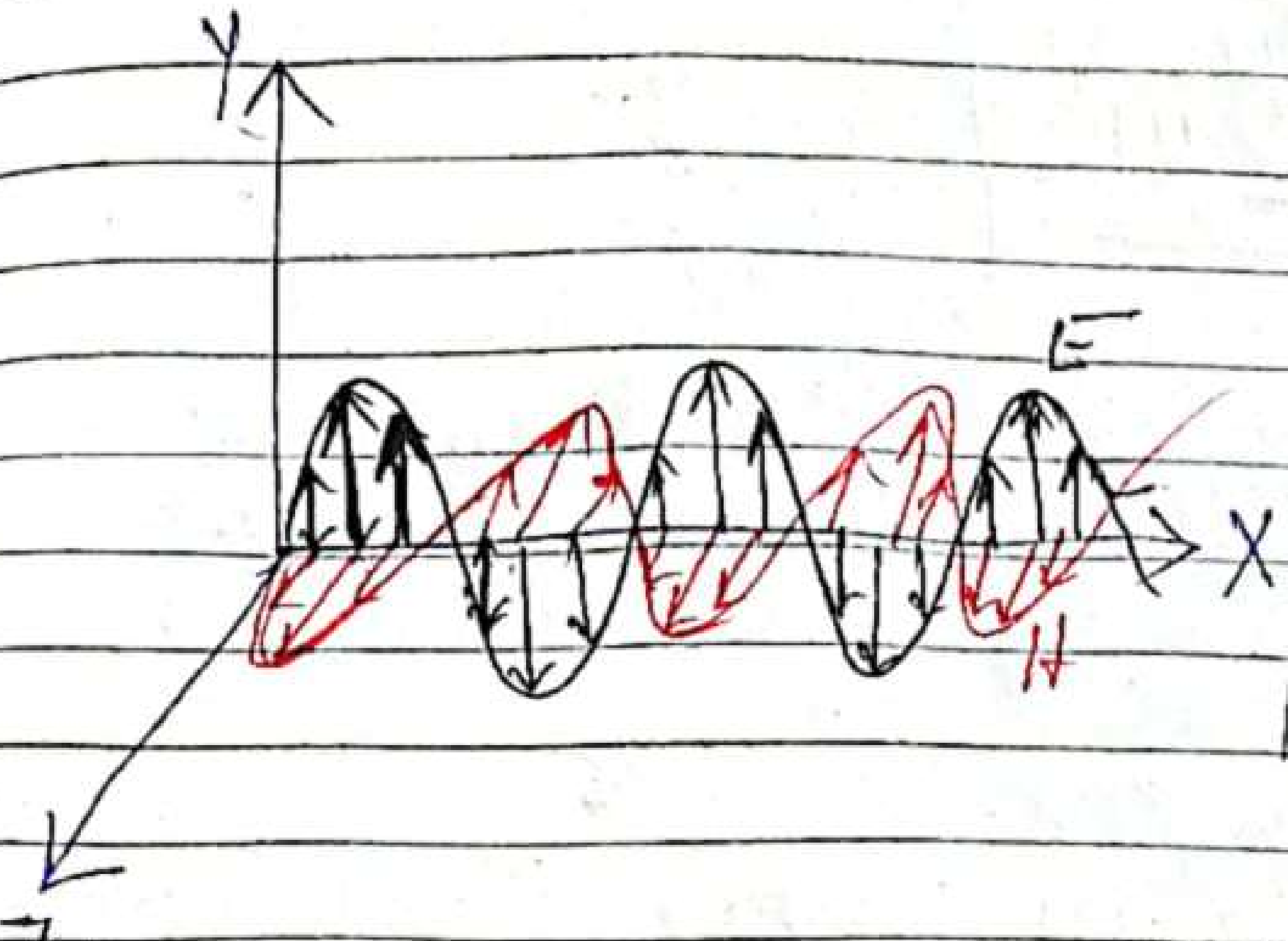
Electromagnetic Energy Density.



$X \Rightarrow$ direction of propagation of EM wave

$Y \Rightarrow$ direction of electric field E

$Z \Rightarrow$ direction of magnetic field H.

As EM wave consists of electric and magnetic field simultaneously, so the energy must be stored in these two fields.

According to definition,

$$\text{energy density} = \frac{\text{Total energy stored}}{\text{volume.}}$$

= sum of electric field energy density + sum of magnetic field energy density.

Now, electric field energy density $u_E = \frac{1}{2}\vec{E}\cdot\vec{D}$,

where $\vec{E}$ = electric field

$\vec{D}$ = electric displacement = $\epsilon\vec{E}$,

where ϵ = permittivity of the medium

$$\therefore\ u_E = \frac{1}{2}\vec{E}\cdot\vec{D} = \frac{1}{2}\vec{E}\cdot\epsilon\vec{E} = \frac{1}{2}\epsilon(\vec{E}\cdot\vec{E}) = \frac{1}{2}\epsilon E^2$$

$$\therefore\ \boxed{u_E = \frac{1}{2}\epsilon E^2} \quad -(1)$$

Similarly, magnetic field energy density $u_M = \frac{1}{2}\vec{H}\cdot\vec{B}$,

where $\vec{H}$ = magnetic field

$\vec{B}$ = magnetic field intensity = $\mu\vec{H}$,

where μ = permeability of the medium

$$\therefore\ u_M = \frac{1}{2}\vec{H}\cdot\vec{B} = \frac{1}{2}\vec{H}\cdot\mu\vec{H} = \frac{1}{2}\mu(\vec{H}\cdot\vec{H}) = \frac{1}{2}\mu H^2$$

$$\therefore\ \boxed{u_M = \frac{1}{2}\mu H^2} \quad -(2)$$

If u = total energy density of EM wave, then,

$$u = u_E + u_M$$

$$\boxed{u = \frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2} \quad \text{--- (3)}$$

As electric field is along Y- direction and its a simple ~~f~~ sinusoidal wave, so the component of the electric field $\vec{E}$ along Y-axis can be written as,

$$E_y = E_m \cos(kx - \omega t) \quad \text{--- (5)}$$

where $k = 2\pi/\lambda$ = propagation constant or wavenumber.

$\omega = 2\pi f = 2\pi/T$ = angular frequency.

~~EM wave~~

Velocity of electro magnetic wave, $C = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{E}{B}$

$= \frac{E}{B}$,

or, $B = E\sqrt{\mu_0 \epsilon_0}$.

Proof of $C = \frac{E}{B}$

Electric field, $E = F/q$.

We know, for Lorentz force,

$F_L = i \cdot B \cdot l$

$= \frac{q}{t} \cdot B \cdot l$

$= q \cdot B \cdot \frac{l}{t}$

$F_L = q \cdot B \cdot v \Rightarrow B = \frac{F}{qv}$

Now,

From dim. analysis,

$[c] = \frac{[L]}{[T]}$

$\left[\frac{E}{B}\right] = \frac{F/q}{F/qv} = [v] = \frac{[L]}{[T]}$

So, $\boxed{c = \frac{E}{B}}$ Proved

$$u = \frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2.$$

$$= \frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu \cdot \frac{B^2}{\mu^2} \left[\because \vec{B} = \mu\vec{H}\right].$$

$$= \frac{1}{2}\epsilon E^2 + \frac{1}{2}\frac{B^2}{\mu}.$$

$$= \frac{1}{2}\epsilon E^2 + \frac{1}{2\mu} \cdot E^2 \mu\epsilon$$

$$= \frac{1}{2}\epsilon E^2 + \frac{1}{2}\epsilon E^2$$

$$\boxed{u = \epsilon E^2} \quad \text{--- (4)}$$

Eqn (4) gives the energy density in terms of electric field

From (3) → $u = \frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2$

$C = \frac{E}{B} = \frac{1}{\sqrt{\mu\epsilon}}$.

or, $E = \frac{B}{\sqrt{\mu\epsilon}}$.

$u = \frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2$

$= \frac{1}{2}\cancel{\epsilon}\frac{B^2}{\mu\cancel{\epsilon}} + \frac{1}{2}\cancel{\mu}\frac{B^2}{\mu^{\cancel{2}}}$

$= \frac{1}{2}\frac{B^2}{\mu} + \frac{1}{2}\frac{B^2}{\mu} = \frac{B^2}{\mu}$

$\therefore \boxed{u = \frac{B^2}{\mu}}$ → In term of B.

- Average Electromagnetic Energy Density.

For an EM wave, total energy density $u = u_E + u_M$.

$$= \frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2$$

$$= \frac{1}{2}\epsilon E^2 + \frac{1}{2}\frac{B^2}{\mu} \quad \text{--- (5)}$$

Now, Average Electric energy density:

$$\langle u_E \rangle = \left\langle \frac{1}{2}\epsilon E^2 \right\rangle = \frac{1}{2}\epsilon \langle E^2 \rangle \quad \left[\because \frac{1}{2}\epsilon = \text{Const}\right] \quad \text{--- (6)}$$

As electric field is along y-axis, and its a simple sinusoidal wave, so the component of the electric field $\vec{E}$ along y-direction can be written as,

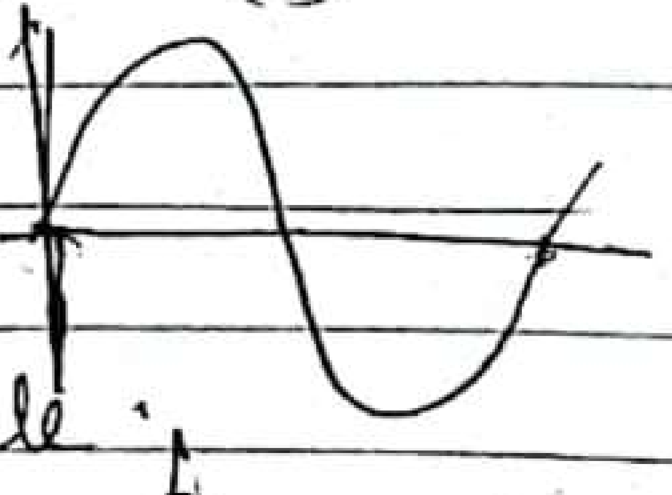
$$E_y = E_m \sin(kx - \omega t)$$

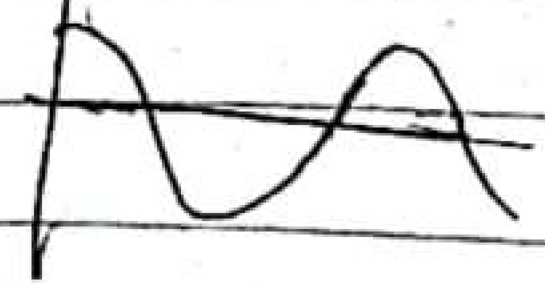
$$\text{or,} \quad E_y = E_m \cos(kx - \omega t).$$

putting this in eqn (6), we get:

$$\langle u_E \rangle = \frac{1}{2}\epsilon \langle E_m^2 \cos^2(kx - \omega t) \rangle \quad \text{--- (7)}$$

We know, $\langle \sin^2\theta \rangle = \langle \cos^2\theta \rangle = 1/2$ & $\langle \sin\theta \rangle = \langle \cos\theta \rangle = 0$ } for a full cycle.





$\therefore$ eqn (7) becomes

$$\langle u_E \rangle = \frac{1}{2}\epsilon \cdot \frac{1}{2}E_m^2$$

$$\boxed{\langle u_E \rangle = \frac{1}{4}\epsilon E_m^2} \quad \text{--- (8)}$$

$$\text{or,} \quad \langle u_E \rangle = \frac{1}{2}\epsilon \frac{E_m^2}{2} = \frac{1}{2}\epsilon \left(\frac{E_m}{\sqrt{2}}\right)^2 = \frac{1}{2}\epsilon E_{r.m.s}^2$$

$\therefore \boxed{\langle u_E \rangle = \frac{1}{2}\epsilon E_{r.m.s}^2}$ --- (9) Eqn (8) & (9) give Average Electric Energy Density.

Similarly, Average Magnetic Energy Density,

$$\langle u_M \rangle = \frac{1}{2}\mu \langle H^2 \rangle = \frac{1}{2}\mu \left\langle \frac{B^2}{\mu^2} \right\rangle = \frac{1}{2\mu}\langle B^2 \rangle$$

As the magnetic field is along Z- direction and its a simple sinusoidal wave,

$$\therefore B_Z = B_m \sin(kx - \omega t)$$

$$\therefore \langle u_M \rangle = \frac{1}{2\mu} \langle B_m^2 \sin^2(kx - \omega t) \rangle$$

$$= \frac{B_m^2}{2\mu} \langle \sin^2(kx - \omega t) \rangle$$

$$= \frac{B_m^2}{2\mu} \cdot \frac{1}{2} = \frac{1}{4} \frac{B_m^2}{\mu}$$

$$\boxed{\langle u_M \rangle = \frac{1}{4\mu} B_m^2} \quad -(10)$$

$$\langle u_M \rangle = \frac{1}{2\mu}\left(\frac{B_m^2}{2}\right) = \frac{1}{2\mu}\left(\frac{B_m}{\sqrt{2}}\right)^2 = \frac{1}{2\mu} B_{r.m.s}^2$$

$$\boxed{\langle u_M \rangle = \frac{1}{2\mu} B_{r.m.s}^2} \quad -(11)$$

Eqns (10) & (11) give Average Magnetic Energy Density.

• Prove

Avergy Electric energy Density } = Average Magnetic energy Density.

$\langle u_E \rangle = \langle u_M \rangle$

We know, $\langle u_E \rangle = \frac{1}{4} \epsilon E_m^2 = \frac{1}{2} \epsilon E_{r.m.s}^2$

& $\langle u_M \rangle = \frac{1}{4\mu} B_m^2 = \frac{1}{2\mu} B_{r.m.s}^2$

Now, $\langle u_E \rangle = \frac{1}{4} \epsilon E_m^2 = \frac{1}{4} \epsilon (cB_m)^2$ $\left[\because c = \frac{E}{B}\right]$

$= \frac{1}{4} \epsilon c^2 B_m^2 = \frac{1}{4} \frac{1}{\mu} B_m^2$ $\left[c = \frac{1}{\sqrt{\mu\epsilon}}\right.$

$= \frac{1}{4\mu} B_m^2$ or, $c^2 = \frac{1}{\mu\epsilon}$

$= \langle u_M \rangle$ or, $\left.\epsilon c^2 = \frac{1}{\mu}\right]$

$\therefore \boxed{\langle u_E \rangle = \langle u_M \rangle}$