University of Calcutta
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PHYSICS
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ELECTRODYNAMICS : DECAY OF CHARGE IN CONDUCTING MEDIUM

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REVISION OF PREVIOUS TOPICS :

MAXWELL'S EQUATION OF ELECTROMAGNETISM

$$\nabla \bullet \mathrm{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \bullet \mathrm{B} = 0$$

$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}$$

$$\nabla \times \mathrm{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathrm{B}}{\partial t} \right)$$

WAVE EQUATION DERIVATION FROM MAXWELL'S EQUATION

Taking curl of 3rd equation:

$$\nabla \times \nabla \times E = -\mu \frac{\partial}{\partial t}(\nabla \times H)$$

Using 4th equation:

$$\nabla \times \nabla \times E = -\mu \frac{\partial}{\partial t}\left(\sigma E + \varepsilon \frac{\partial E}{\partial t} \right)$$

$$\nabla(\nabla . E) - \nabla^2 E = -\sigma\mu \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2}$$

From equation 1st: $\nabla . E = 0$

$$\boxed{\nabla^2 E - \sigma\mu \frac{\partial E}{\partial t} - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0}$$

For Electric Field **E**

PLANE MONOCHROMATIC WAVE IN A CONDUCTING MEDIUM

Recall that in a "simple" dielectric material, we derived the wave equations:

$$\nabla^2 \vec{E} - \mu\varepsilon \ddot{\vec{E}} = 0 \tag{1}$$

$$\nabla^2 \vec{B} - \mu\varepsilon \ddot{\vec{B}} = 0 \tag{2}$$

To derive these equations, we used Maxwell's equations with the assumptions that the charge density ρ and current density J were zero, and that the permeability μ and permittivity ε were constants.

We found that the above equations had plane-wave solutions, with phase velocity:

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \tag{3}$$

Maxwell's equations imposed additional constraints on the directions and relative amplitudes of the electric and magnetic fields.

How are the wave equations (and their solutions) modified for the case of electrically conducting media?

We shall restrict our analysis to the case of ohmic conductors, which are defined by:

$$\vec{J} = \sigma \vec{E} \tag{4}$$

where σ is a constant, the conductivity of the material.

All we need to do is substitute from equation (4) into Maxwell's equations, then proceed as for the case of a dielectric...

In our "simple" conductor, Maxwell's equations take the form:

$$\nabla \cdot \vec{E} = 0 \tag{5}$$

$$\nabla \cdot \vec{B} = 0 \tag{6}$$

$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{7}$$

$$\nabla \times \vec{B} = \mu\varepsilon\dot{\vec{E}} + \mu\vec{J} \tag{8}$$

where $\vec{J}$ is the current density. Assuming an ohmic conductor, we can write:

$$\vec{J} = \sigma\vec{E} \tag{9}$$

so equation (8) becomes:

$$\nabla \times \vec{B} = \mu\varepsilon\dot{\vec{E}} + \mu\sigma\vec{E} \tag{10}$$

Taking the curl of equation (7) and making appropriate substitutions as before, we arrive at the wave equation:

$$\nabla^2\vec{E} - \mu\sigma\dot{\vec{E}} - \mu\varepsilon\ddot{\vec{E}} = 0 \tag{11}$$

The wave equation for the electric field in a conducting material is (11):

$$\nabla^2\vec{E} - \mu\sigma\dot{\vec{E}} - \mu\varepsilon\ddot{\vec{E}} = 0 \qquad (12)$$

Let us try a solution of the same form as before:

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \qquad (13)$$

Remember that to find the physical field, we have to take the real part. Substituting (13) into the wave equation (11) gives the dispersion relation:

$$-\vec{k}^2 - j\omega\mu\sigma + \omega^2\mu\varepsilon = 0 \qquad (14)$$

Compared to the dispersion relation for a dielectric, the new feature is the presence of an imaginary term in σ. This means the relationship between the wave vector $\vec{k}$ and the frequency ω is a little more complicated than before.

From the dispersion relation (14), we can expect the wave vector $\vec{k}$ to have real and imaginary parts. Let us write:

$$\vec{k} = \vec{\alpha} - j\vec{\beta} \tag{15}$$

for parallel real vectors $\vec{\alpha}$ and $\vec{\beta}$.

Substituting (15) into the dispersion relation (14) and taking real and imaginary parts, we find:

$$\alpha = \omega\sqrt{\mu\varepsilon}\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}}\right]^{1/2} \tag{16}$$

and:

$$\beta = \frac{\omega\mu\sigma}{2\alpha} \tag{17}$$

Equations (16) and (17) give the real and imaginary parts of the wave vector $\vec{k}$ in terms of the frequency ω, and the material properties μ, ε and σ.

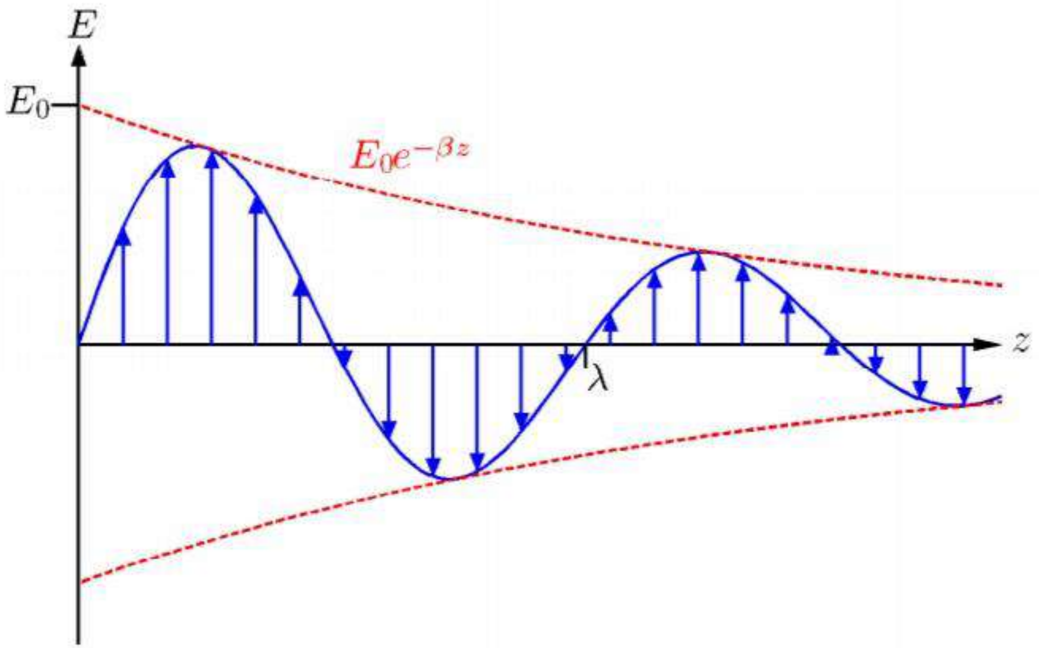
Using equation (15) the solution (13) to the wave equation in a conducting material can be written:

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \vec{\alpha}\cdot\vec{r})} e^{-\vec{\beta}\cdot\vec{r}} \tag{18}$$

The first exponential factor, $e^{j(\omega t - \vec{\alpha}\cdot\vec{r})}$ gives the usual plane-wave variation of the field with position $\vec{r}$ and time t; note that the conductivity of the material affects the wavelength for a given frequency.

The second exponential factor, $e^{-\vec{\beta}\cdot\vec{r}}$ gives an exponential decay in the amplitude of the wave...

Plane Monochromatic Wave in a Conducting Material



In a "simple" non-conducting material there is no exponential decay of the amplitude: electromagnetic waves can travel for ever, without any loss of energy.

If the wave enters an electrical conductor, however, we can expect very different behaviour. The electrical field in the wave will cause currents to flow in the conductor. When a current flows in a conductor (assuming it is not a superconductor) there will be some energy changed into heat. This energy must come from the wave. Therefore, we expect the wave gradually to decay.

The varying electric field must have a magnetic field associated with it. Presumably, the magnetic field has the same wave vector and frequency as the electric field: this is the only way we can satisfy Maxwell's equations for all positions and times. Therefore, we try a solution of the form:

$$\vec{B}(\vec{r},t) = \vec{B}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \tag{19}$$

Now we use Maxwell's equation (7):

$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{20}$$

which gives:

$$\vec{k} \times \vec{E}_0 = \omega \vec{B}_0 \tag{21}$$

or:

$$\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0 \tag{22}$$

The magnetic field in a wave in a conducting material is related to the electric field by (22):

$$\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0 \tag{23}$$

As in a non-conducting material, the electric and magnetic fields are perpendicular to the direction of motion (the wave is a transverse wave) and are perpendicular to each other.

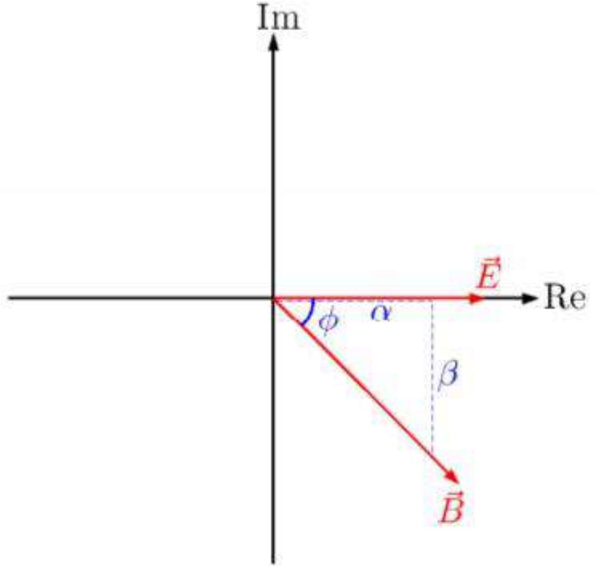
But there is a new feature, because the wave vector is complex.

In a non-conducting material, the electric and magnetic fields were in phase: the expressions for the fields both had the same phase angle ϕ_0. In complex notation, the complex phase angles of the field amplitudes $\vec{E}_0$ and $\vec{B}_0$ were the same.

In a conductor, the complex phase of $\vec{k}$ gives a phase difference between the electric and magnetic fields.

In a conducting material, there is a difference between the phase angles of $\vec{E}_0$ and $\vec{B}_0$, given by the phase angle ϕ of $\vec{k}$. This is:

$$\tan\phi = \frac{\beta}{\alpha} \tag{24}$$



Let us consider the special case of a very good conductor. In this case:

$$\sigma \gg \omega\varepsilon \tag{29}$$

From equation (16), we then have:

$$\alpha \approx \sqrt{\frac{\omega\mu\sigma}{2}} \tag{30}$$

and from equation (17) we have:

$$\beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} \approx \alpha \tag{31}$$

In the case of a very good conductor, the real and imaginary parts of the wave vector $\vec{k}$ become equal. This means that the decay of the wave is very fast in terms of the number of wavelengths.

Note that the vectors $\vec{\alpha}$ and $\vec{\beta}$ have the same units as $\vec{k}$, i.e. meters^{-1}.

The electric field in the wave varies as (18):

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \vec{\alpha}\cdot\vec{r})} e^{-\vec{\beta}\cdot\vec{r}} \tag{32}$$

The phase velocity is the velocity of a point that stays in phase with the wave. Consider a wave moving in the $+z$ direction:

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \alpha z)} e^{-\beta z} \tag{33}$$

For a point staying at a fixed phase, we must have:

$$\omega t - \alpha z(t) = \text{constant} \tag{34}$$

So the phase velocity is given by:

$$v_p = \frac{dz}{dt} = \frac{\omega}{\alpha} \tag{35}$$

But note that in a good conductor, α is itself a function of ω...

For a poor conductor ($\sigma \ll \omega\varepsilon$), we have:

$$\alpha \approx \omega\sqrt{\mu\varepsilon} \tag{36}$$

so the phase velocity in a poor conductor is:

$$v_p = \frac{\omega}{\alpha} \approx \frac{1}{\sqrt{\mu\varepsilon}} \tag{37}$$

If μ and ε are constants (i.e. are independent of ω) then the phase velocity is independent of the frequency: there is no dispersion.

However, in a good conductor ($\sigma \gg \omega\varepsilon$), we have:

$$\alpha \approx \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\mu\varepsilon}\sqrt{\frac{\omega\sigma}{2\varepsilon}} \tag{38}$$

Then the phase velocity is given by:

$$v_p = \frac{\omega}{\alpha} \approx \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\frac{2\omega\varepsilon}{\sigma}} \tag{39}$$

The phase velocity depends on the frequency: there is dispersion!

The presence of dispersion means that the group velocity v_g (the velocity of a wave pulse) can differ from the phase velocity v_p (the velocity of a point staying at a fixed phase of the wave).

To understand what this means, consider the superposition of two waves with equal amplitudes, both moving in the $+z$ direction, and with similar wave numbers:

$$E_x = E_0 \cos\left(\omega_+ t - [k_0 + \Delta k]\, z\right) + E_0 \cos\left(\omega_- t - [k_0 - \Delta k]\, z\right) \tag{40}$$

Using a trigonometric identity:

$$\cos A + \cos B \equiv 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \tag{41}$$

the electric field can be written:

$$E_x = 2E_0 \cos\left(\omega_0 t - k_0 z\right) \cos\left(\Delta\omega\, t - \Delta k\, z\right) \tag{42}$$

where:

$$\omega_0 = \frac{1}{2}\left(\omega_+ + \omega_-\right) \qquad \Delta\omega = \omega_+ - \omega_- \tag{43}$$

We have written the total electric field in our superposed waves as (42):

$$E_x = 2E_0 \cos(\omega_0 t - k_0 z) \cos(\Delta\omega\, t - \Delta k\, z) \tag{44}$$

Assuming that $\Delta k \ll k_0$, the first trigonometric factor represents a wave of (short) wavelength $2\pi/k_0$ and phase velocity:

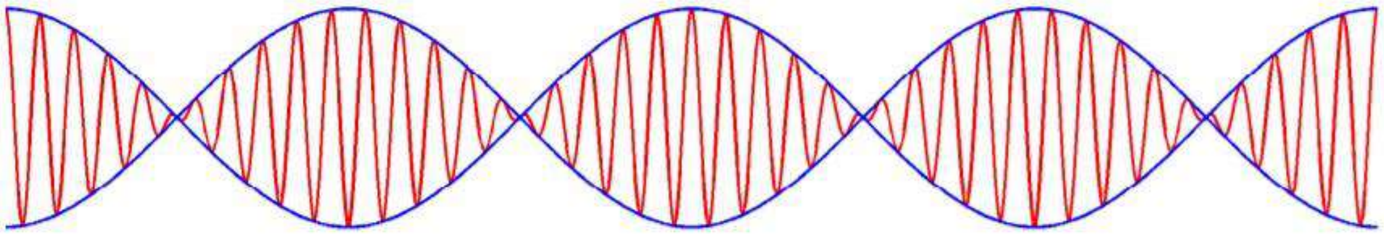
$$v_p = \frac{\omega_0}{k_0} \tag{45}$$

while the second trigonometric factor represents a modulation of (long) wavelength $2\pi/\Delta k$, which travels with velocity:

$$v_g = \frac{\Delta\omega}{\Delta k} \tag{46}$$

v_g is called the group velocity. Since $\Delta\omega$ represents the change in frequency that corresponds to a change Δk in wave number, we can write:

$$v_g = \frac{d\omega}{dk} \tag{47}$$



The red wave moves with the phase velocity v_p; the modulation (represented by the blue line) moves with group velocity v_g.

Since the energy in a wave depends on the local amplitude of the wave, the energy in the wave is carried at the group velocity v_g.

If there is no dispersion, then the phase velocity is independent of frequency:

$$v_p = \frac{\omega}{k} = \text{constant} \tag{48}$$

and the group velocity is equal to the phase velocity:

$$v_g = \frac{d\omega}{dk} = v_p \tag{49}$$

In the absence of dispersion, a modulation resulting from the superposition of two waves with similar frequencies will travel at the same speed as the waves themselves.

However, if there is dispersion, then the group velocity can differ from the phase velocity...

The dispersion relation for an electromagnetic wave in a good conductor is, from (38):

$$\omega = \frac{1}{\mu\varepsilon}\frac{2\varepsilon}{\sigma}\alpha^2 \tag{50}$$

where α is the real part of the wave vector. The group velocity is then:

$$\begin{aligned} v_g &= \frac{d\omega}{d\alpha} \\ &\approx \frac{1}{\mu\varepsilon}\frac{4\varepsilon}{\sigma}\alpha \\ &\approx \frac{2}{\sqrt{\mu\varepsilon}}\sqrt{\frac{2\omega\varepsilon}{\sigma}} \end{aligned} \tag{51}$$

Comparing with equation (39) for the phase velocity of an electromagnetic wave in a good conductor, we find that:

$$v_g \approx 2v_p \tag{52}$$

In other words, the group velocity is approximately twice the phase velocity.

The real part, α, of the wave vector k in a conductor gives the wavelength of the wave. β measures the distance that the wave travels before its amplitude falls to $1/e$ of its original value. Let us write the solution (18) for a wave travelling in the z direction in a good conductor as:

$$\vec{E}(\vec{r},t) = \vec{E}_0'(\vec{r})e^{j(\omega t - \vec{\alpha}\cdot\vec{r})} \tag{53}$$

where:

$$\vec{E}_0'(\vec{r}) = \vec{E}_0 e^{-\vec{\beta}\cdot\vec{r}} \tag{54}$$

The amplitude of the wave falls by a factor $1/e$ in a distance $1/\beta$. We define the *skin depth* δ:

$$\delta = \frac{1}{\beta} \tag{55}$$

From equation (31), we see that for a good conductor ($\sigma \gg \omega\varepsilon$), the skin depth is given by:

$$\delta \approx \sqrt{\frac{2}{\omega\mu\sigma}} \tag{56}$$

For example, consider silver, which has conductivity $\sigma \approx 6.30 \times 10^7\,\Omega^{-1}\mathrm{m}^{-1}$, and permittivity $\varepsilon \approx \varepsilon_0 \approx 8.85 \times 10^{-12}\,\mathrm{Fm}^{-1}$.

For radiation of frequency 10^{10} Hz, the "good conductor" condition is satisfied, and the skin depth of the radiation is approximately 0.6 micron (0.6×10^{-6} m).

Note that in vacuum, the wavelength of radiation of frequency 10^{10} Hz is about 3 cm; but in silver, the wavelength is:

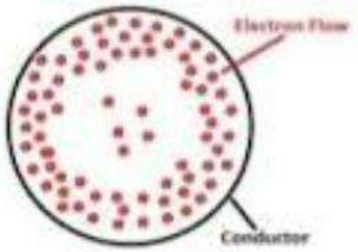
$$\lambda = \frac{2\pi}{\alpha} \approx 2\pi\delta \approx 4\,\mathrm{micron} \tag{57}$$

CLEAR YOUR CONCEPT ABOUT

SKIN EFFECT/ SKIN DEPTH

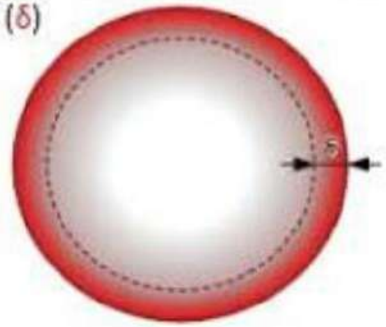
What is Skin Effect?

- **Skin effect** is the tendency of an alternating electric current (AC) to become distributed within a conductor such that the current density is largest near the surface of the conductor, and decreases with greater depths in the conductor.

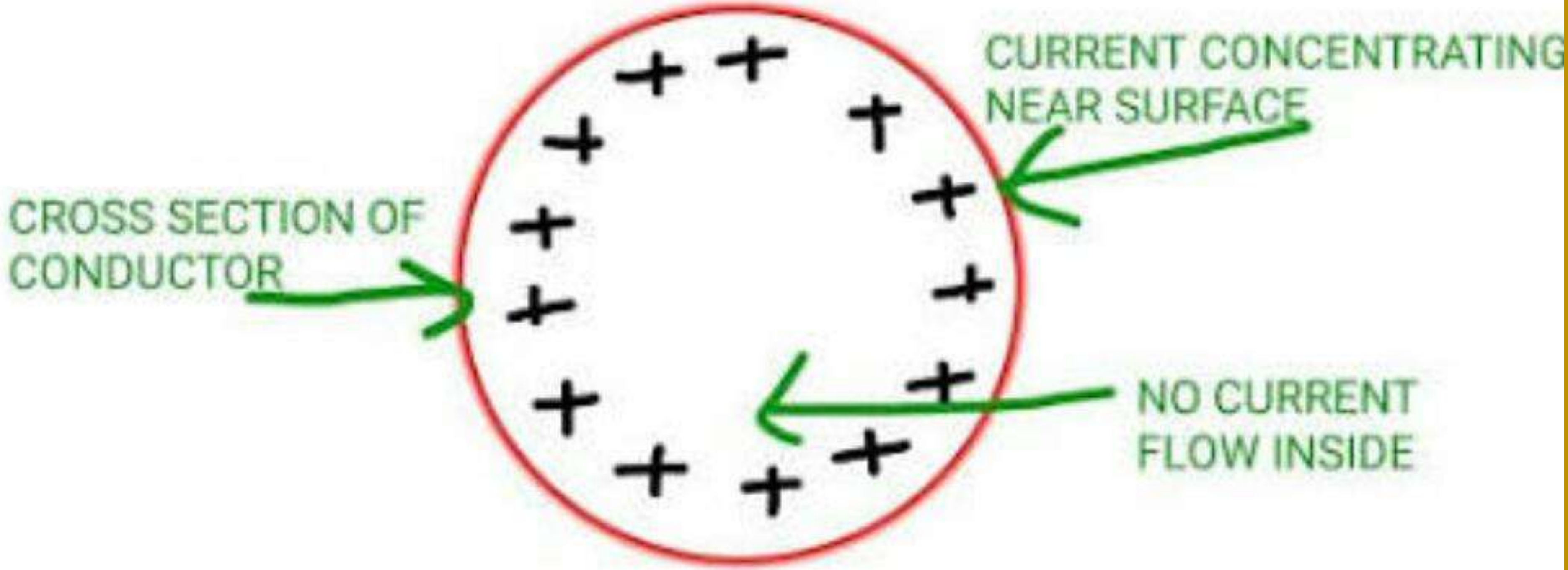


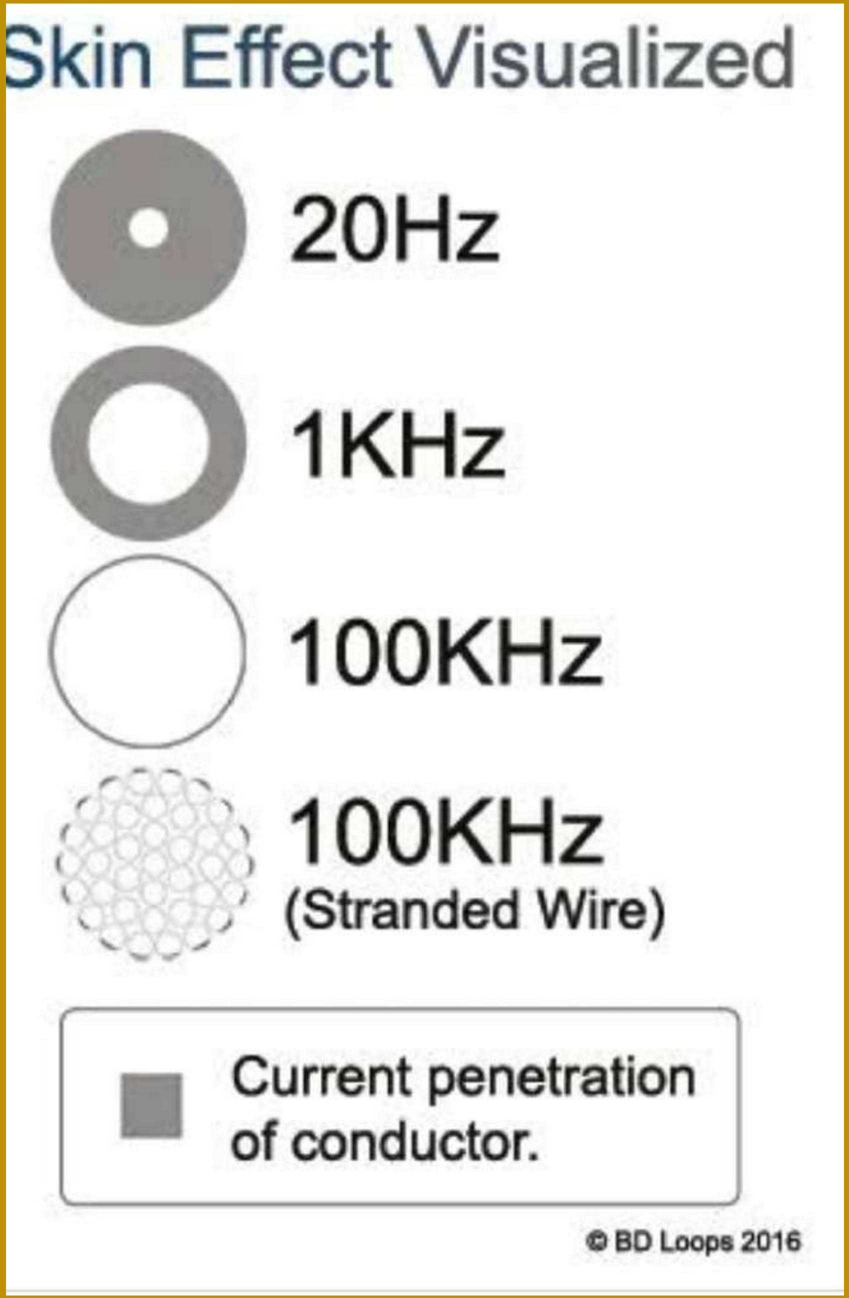
Skin Depth(δ)

- *Skin depth* is a measure of the depth at which the current density falls to 1/e of its value near the surface
- Skin depth also describes the exponential decay of the electric and magnetic fields, as well as the density of induced currents
- Distribution of current flow in a cylindrical conductor, For alternating current, most (63%) of the electric current flows between the surface and the skin depth (δ)



WHAT IS SKIN EFFECT?





As SKIN DEPTH is inversely proportional to the square root of the FREQUENCY, it is clear from the picture that
' as frequency increases skin depth decreases'

SOLVED PROBLEMS -1

2. *Find the values of E and H on the surface of a wire carrying a current. Show, by computing the Poynting vector, that it represents a flow of energy into the wire.* (C.U. 1982)

Ans. Let a constant current I flow through the wire of radius a and length l, as shown in Fig.16.6. If V_d is the potential difference across the length l of the wire, the electric field on the surface of the wire is directed along its length and is given by

$$E = \frac{V_d}{l}.$$

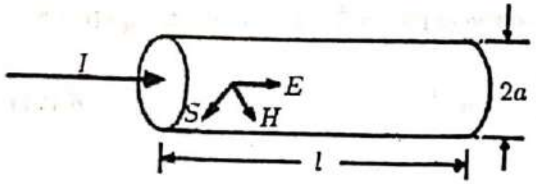


Fig.16.6. Application of Poynting's theorem to current flow in a wire

The magnetic field H at the surface is directed along the tangent to the curved surface, and from Ampere's circuital law, it is given by

$$H = \frac{I}{2\pi a}.$$

The Poynting vector given by

$$\vec{S} = \vec{E} \times \vec{H},$$

is directed radially into the wire, as shown in Fig.16.6.

Since the fields are steady, the rate of change of electromagnetic energy is zero. Therefore the conservation of energy expressed by Poynting's theorem [Eq.(16.17)] gives $-\oint_{S_0} \vec{S}\cdot\hat{n}\, dS_0 = \int_V \vec{J}\cdot\vec{E} dV$. The rate of flow of energy into the wire through the closed surface of radius a and length l is

$$-\oint_{S_0} \vec{S}\cdot\hat{n}\, dS_0 = \oint_{S_0} EH\, dS_0 = EH(2\pi a l) = V_d I, \qquad (i)$$

on substituting the values of E and H.

The volume integral $\int_V \vec{E}\cdot\vec{J} dV$ is calculated as follows :

Since $\vec{J} = \sigma_c \vec{E}$, we have

$$\int_V \vec{E}\cdot\vec{J} dV = \int_V \sigma_c E^2 dV = \sigma_c E^2(\pi a^2 l) = V_d^2\left(\frac{\sigma_c \pi a^2}{l}\right),$$

on substituting for E. The resistance R of the wire is $R = l/(\sigma_c \pi a^2)$. Hence

$$\int_V \vec{E}\cdot\vec{J} dV = \frac{V_d^2}{R} = V_d I. \qquad (ii)$$

From (i) and (ii) we find that Poynting's theorem is satisfied.

The physical interpretation of this result is that the electromagnetic energy flows into the wire through its curved surface from outside. As the voltage source produces the field, the electromagnetic energy appears to flow from the voltage source into the outer space. It then enters the wire and is dissipated as heat.

SOLVED PROBLEMS -2

4. *Find from Poynting flow the mean value of the intensity of the magnetic field in air at a distance of* 100 *cm from a radiating source of power* 10 *kW.* (C.U. 1987)

Ans. Considering the radiating source as a point source, the total energy flow over a sphere with the source at the centre is $10\,\text{kW} = 10^4$ J/s. At the distance of 100 cm(= 1 m), the surface area of the sphere is $S_0 = 4\pi 1^2 = 4\pi$ m^2. The energy flow per unit area per second is $10^4/S_0 = 10^4/(4\pi)$ J.

From Poynting's theorem, the energy flow per unit area per second is

$$S = \frac{1}{2}EH,$$

where E is the electric field amplitude of the wave and H is the magnetic field amplitude of the wave, E and H being perpendicular. Now, $H = \sqrt{\epsilon_0/\mu_0}E$ or, $E = \sqrt{\frac{\mu_0}{\epsilon_0}}H = 377H$, since $\sqrt{\mu_0/\epsilon_0}$ = intrinsic impedance of free space = 377 Ω. Therefore

$$S = \frac{1}{2}(377)H^2 = \frac{10^4}{4\pi}$$

$$\text{or,} \quad H = \frac{10^2}{\sqrt{2\pi \times 377}} = 2.05\ \text{A/m}.$$

The r.m.s. value of the magnetic intensity is $H/\sqrt{2} = 1.44$ A/m.

5. A TEM wave of f...

ASSIGNMENTS

3. State and prove Poynting's theorem. **(C.U. 1982)**

4. (a) A plane electromagnetic wave is incident normally on a metal of electrical conductivity σ. Show that the electromagnetic wave is damped inside the conductor and find the skin depth. **(C.U. 1982)**

8. (a) Write down Maxwell's electromagnetic field equations and explain the physical significance of each. **(C.U. 1988, cf. Burd. U. 1996)**

(b) Show how Maxwell's equations in free space imply local conservation of charge (continuity equation) **(C.U. 1994)**

9. (a) Explain the concept of displacement current and show its importance.

(cf. C.U. 1989, Burd. U. 1995)

Acknowledge

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Electricity and Magnetism

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