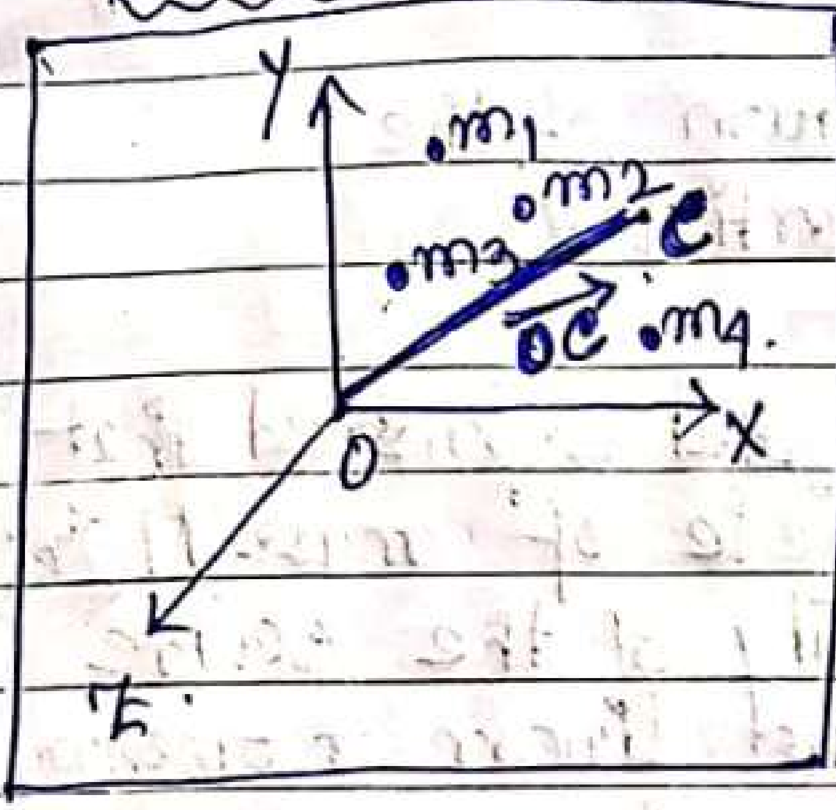
LINEAR MOMENTUM OF A SYSTEM OF PARTICLES



Acc. to definition,

$$\boxed{\text{Position Vector} = \frac{\text{Linear Moment.}}{\text{Mass}}}$$

Considering a point c, the position vector of which relative to a fixed point O is given by,

$$\boxed{\vec{OC} = \frac{\sum m_i \vec{r_i}}{\sum m_i}, \text{ where } i = 1 \text{ to } n} \quad \text{— (4) (previous note).}$$

Now, $\sum_{i=1}^{n} m_i = M$
= total mass of the syst of particles.

Viewing eqn (4), we can also express,

$$\text{Position Vector} = \frac{\text{Linear Momentum}}{\text{Mass}}$$

or, $$\boxed{\vec{V} = \frac{\sum m_i \vec{v_i}}{M}} \quad \text{— (9).}$$

where $\sum_{i=1}^{n} m_i \vec{v_i}$ = mass x velocity
= momentum $= \sum \vec{p_i}$

eqn (9) $\longrightarrow M\vec{V} = \sum_{i=1}^{n} m_i \vec{v_i} = \sum_{i=1}^{n} \vec{p_i}$

The term $\vec{V}$ can be defined as the velocity of the centre of mass.

$$\boxed{M\vec{V} = \sum \vec{p_i}} \quad \text{---(10)}$$

= linear momentum of the system of particles

From eqn (10), it can be ~~stat~~ expressed that — 'if a fictitious particle of mass $M(\sum m_i)$ moves with a velocity of the centre of mass $\vec{V}(\sum v_i)$, its linear momentum will be equal to the linear momentum of the system of particles.

If $\vec{P}$ = linear momentum of the system of particles,

then eqn (10) can be written as,

$$\boxed{\vec{P} = M\vec{V}} \quad \text{---(11)}$$

Law of Conservation of Linear Momentum.

Eqn. (11) $\rightarrow \vec{P} = M\vec{V}$

Differentiating $\vec{P} \Rightarrow \frac{d\vec{P}}{dt} = M\frac{d\vec{V}}{dt}$

$$= \vec{F}$$

or, $\frac{d\vec{P}}{dt} = \vec{F}$

Change in momentum = Force.

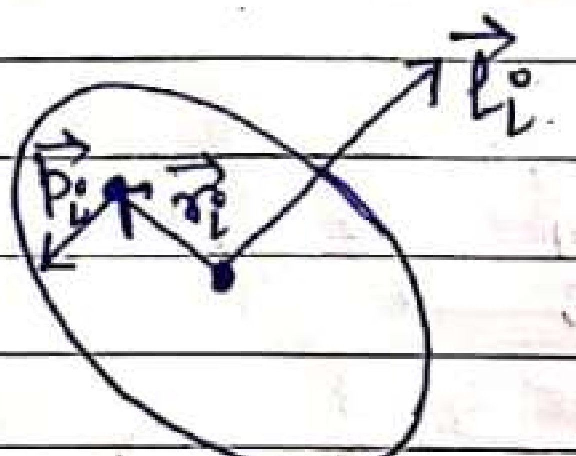
If $\vec{F} = 0$, $\frac{d\vec{P}}{dt} = 0$ i.e. $\vec{P}$ = const.

In absence of force, rate of change in momentum will be zero i.e the linear momentum of the system of particles remains const — This is Law of Conservation of Linear Momentum

Angular Momentum of a System of Particles

Acc. to definition,

Angular Momentum = position vector x Linear Momentum



Let us consider a system of particles moving about an arbitrary fixed point O as origin.

$\vec{r_i}$ = position vector of ith particle from O
$\vec{p_i}$ = linear momentum " "

Then acc. to definition, the angular momentum of the ith particle

$$\boxed{\vec{l_i} = \vec{r_i} \times \vec{p_i}} \quad \text{--- (12).}$$

The total angular momentum of a system of particles referred to the origin O is given by,

$$\vec{L} = \sum_{i=1}^{n} \vec{l_i} = \sum_{i=1}^{n} \vec{r_i} \times \vec{p_i} \quad \text{--- (13)}.$$

eqn.(13) $-\sum_i \vec{OC} \times \vec{p_i} + \sum_i \vec{OC} \times \vec{p_i}$

$$\vec{L} = \sum \vec{r_i} \times \vec{p_i} - \sum \vec{OC} \times \vec{p_i} + \sum \vec{OC} \times \vec{p_i}$$

$$\boxed{\vec{L} = \sum (\vec{r_i} - \vec{OC}) \times \vec{p_i} + \sum \vec{OC} \times \vec{p_i}} \quad \text{--- (14).}$$

where $\vec{OC}$ = position vector of the mass centre relative to the origin O

$\vec{r_i}'$ = position vector of ith particle relative to COM.
put $= (\vec{r_i} - \vec{OC})$,

eqn (14) $\rightarrow$ $\boxed{\vec{L} = \sum \vec{r_i}' \times \vec{p_i} + \sum \vec{OC} \times \vec{p_i}}$ — (15).

$\sum \vec{r_i}' \times \vec{p_i}$ = total angular momentum about COM
$= \vec{L}'$

eqn (15) $\rightarrow$ $\vec{L} = \vec{L}' + \sum \vec{OC} \times \vec{p_i}$
$= \vec{L}' + \vec{OC} \times \sum \vec{p_i}$
~~then~~ -
$= \vec{L}' + \vec{OC} \times \vec{P}$, where $\vec{P} = \sum \vec{p_i}$.

$\therefore$ $\boxed{\vec{L} = \vec{L}' + (\vec{OC} \times \vec{P})}$ — (16).

eqn (16), $\vec{OC} \times \vec{P}$ = angular momentum of the COM about origin

EXTERNAL & INTERNAL TORQUES

The total angular momentum of a system of particles referred to the origin O is given by,

$$\vec{L} = \sum \vec{r_i} \times \vec{p_i} \quad \text{—} (13)$$

Differentiating eqn (13),

$$\frac{d\vec{L}}{dt} = \frac{d}{dt} \sum \vec{r_i} \times \vec{p_i}$$

$$= \sum_i \frac{d\vec{r_i}}{dt} \times \vec{p_i} + \sum \vec{r_i} \times \frac{d\vec{p_i}}{dt}$$

$$= \sum_i \vec{v_i} \times \vec{p_i} + \sum \vec{r_i} \times \frac{d\vec{p_i}}{dt} \quad \text{—} (17),$$

where $\vec{v_i} = \frac{d\vec{r_i}}{dt}$
= velocity

We know, $\vec{p_i}$ = momentum
= mass × velocity
$= m_i \vec{v_i}$

eqn (17) → $\frac{d\vec{L}}{dt} = \sum \vec{v_i} \times m_i \vec{v_i} + \sum \vec{r_i} \times \frac{d\vec{p_i}}{dt}$

$$= \sum m_i (\vec{v_i} \times \vec{v_i}) + \sum \vec{r_i} \times \frac{d\vec{p_i}}{dt}$$

$$= 0 + \sum \vec{r_i} \times \frac{d\vec{p_i}}{dt} \quad [\because \vec{v_i} \times \vec{v_i} = 0]$$

$$\therefore \frac{d\vec{L}}{dt} = \sum \vec{r_i} \times \frac{d\vec{p_i}}{dt}$$

$$\boxed{\frac{d\vec{L}}{dt} = \sum \vec{r_i} \times \vec{F_i}} \quad \text{—} (18),$$

where $\vec{F_i} = \frac{d\vec{p_i}}{dt} = \frac{\text{change in momentum}}{\text{change in time}}$

= Force.

eqn (18) $\Rightarrow$.

$$\left.\begin{matrix}\text{Change in} \\ \text{angular Momentum}\end{matrix}\right\} = \left\{\begin{matrix}\text{position} \\ \text{vector}\end{matrix}\right\} \times \text{Force}.$$

$$= \text{torque}$$

$\therefore$ eqn (18) $\rightarrow$ $\boxed{\frac{d\vec{L}}{dt} = \sum \vec{r_i} \times \vec{F_i} = N_T}$ — (19),

where N_T = total torque about O.

- In a system of particle, all particles exert force to each other. ~~So one partic~~
- So every particle experiences force from all other particle.
- So we can say that the force on the i^{th} particle from all other particles j is

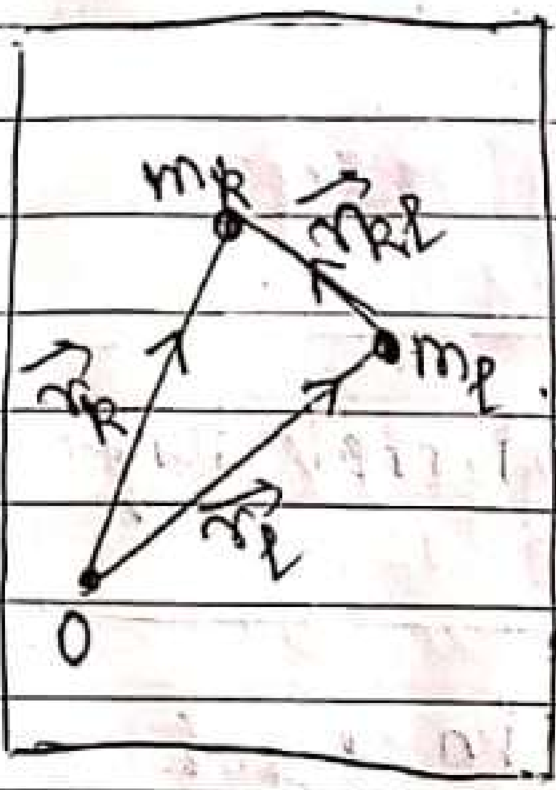
$$\sum_{\substack{j=1 \\ i \neq j}}^{n} \vec{F}_{ij}$$

- This force F_{ij} gives rise the internal force of the system of particle.

Acc. to definitⁿ, Torque = position × Force.

then Internal Torque = position vector × Internal Force

$$\vec{N}_{internal} = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i\neq j}}^{n} \vec{r}_i \times \vec{F}_{ij} \quad -(20)$$



Acc. to the Fig on L.H.S.

$\vec{r}_k$ = position vector of m_k from O

$\vec{r}_l$ = " " " m_l " "

$\vec{r}_{kl}$ = relative position vector betⁿ m_k & m_l

eqn (20) → In the R.H.S. of this expression, we get pair of terms like

$$(\vec{r}_k \times \vec{F}_{kl} + \vec{r}_l \times \vec{F}_{lk})$$

which may also be written as,

$$(\vec{r}_k \times \vec{F}_{kl} - \vec{r}_l \times \vec{F}_{kl}) \quad , \because \vec{F}_{lk} = -\vec{F}_{kl}$$

$$= (\vec{r}_k - \vec{r}_l) \times \vec{F}_{kl}$$

$$= \vec{r}_{kl} \times \vec{F}_{kl}$$

Since dirⁿ of $\vec{r}_{kl}$ = dirⁿ of $\vec{F}_{kl}$

$$\therefore \vec{r}_{kl} \times \vec{F}_{kl} = 0.$$

$$\therefore \boxed{\vec{N}_{internal} = 0} \quad -(21).$$

eqn (19) $\rightarrow$

$$\frac{d\vec{L}}{dt} = N_T = N_{ext} + N_{int}$$

$$= N_{ext} + 0$$

$$\boxed{\frac{d\vec{L}}{dt} = N_{ext}} \quad \text{--- (22)}$$

$$\therefore \boxed{\frac{\text{change in angular momentum}}{\text{change in time}} = \text{External torque}}$$

eqn (22) $\rightarrow$ Time rate of change in angular momentum relative to an arbitrary fixed point is equal to the external torque or the moment of the external force on the system about the same point

Special Case :-

1). If $\vec{N}_{ext} = 0$, then $\frac{d\vec{L}}{dt} = 0$

or, $\vec{L} = \text{Const}$.

In absence of external torque, the angular momentum of the system of particles remains constant — This is the Law of Conservation of Angular Momentum

CENTRE OF MASS

1). It is a point of the system of particles that defines the average position of the system relative to the mass of the object.

2). It is a point where Newton's laws of motion can be applied perfectly, no matter what the shape of the object.

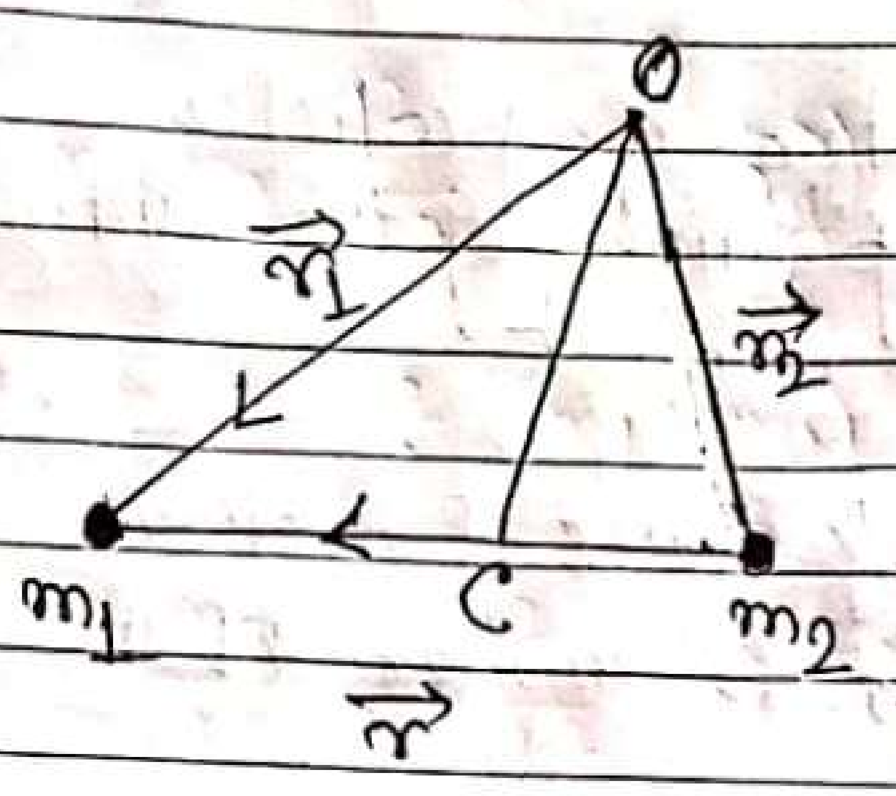
3) COM for regular bodies lies at their geometric centres.

4) COM of the system of particles depends only on the masses of the particles and their relative positions.

5). For normal objects, COM = COG

Centre of mass = Centre of Gravity

TWO BODY PROBLEM



Let us consider two particles of mass m_1 & m_2

$\vec{r}$ = distance of separation of m_1 & m_2

$\vec{r_1}$ = distance of m_1 from the point O

$\vec{r_2}$ = " " m_2 " " " O.

Let $\vec{F_1}$ = external force acted on m_1.

$\vec{F_2}$ = " " " " m_2.

then $\vec{F_{12}}$ = internal force on m_2 for m_1.

$\vec{F_{21}}$ = " " " m_2 " m_2.

Acc. to Newton's Third law,

$$\vec{F_{12}} = -\vec{F_{21}} \quad \text{—(23)}.$$

For m_1, eqn of motion,

$$m_1 \frac{d^2\vec{r_1}}{dt^2} = \vec{F_1} + \vec{F_{12}} \quad \text{—(24)}$$

For m_2, eqn of motion,

$$m_2 \frac{d^2\vec{r_2}}{dt^2} = \vec{F_2} + \vec{F_{21}} \quad \text{—(25)}.$$

• Eqn (24) $\times m_2 \Rightarrow$

$$m_2 \left[m_1 \frac{d^2\vec{r_1}}{dt^2} \right] = m_2 \left[\left(\vec{F_1} \times \vec{F_{12}} \right) \right]$$

$$\text{or, } m_1 m_2 \frac{d^2\vec{r_1}}{dt^2} = m_2 \vec{F_1} \times m_2 \vec{F_{12}} \quad \text{--- (26)}$$

• eqn (25) $\times m_1 \Rightarrow$

$$m_1 \left[m_2 \frac{d^2\vec{r_2}}{dt^2} \right] = m_1 \left[\left(\vec{F_2} \times \vec{F_{21}} \right) \right]$$

$$\text{or, } m_1 m_2 \frac{d^2\vec{r_2}}{dt^2} = m_1 \vec{F_2} \times m_1 \vec{F_{21}} \quad \text{--- (27)}$$

• eqn (26) − eqn (27) $\Rightarrow$

$$m_1 m_2 \frac{d^2\vec{r_1}}{dt^2} - m_1 m_2 \frac{d^2\vec{r_2}}{dt^2}$$

$$= \left(m_2 \vec{F_1} \times m_2 \vec{F_{12}} \right) - \left(m_1 \vec{F_2} \times m_1 \vec{F_{21}} \right)$$

$$\text{or, } m_1 m_2 \frac{d^2}{dt^2}\left(\vec{r_1} - \vec{r_2} \right) = \left(m_2 F_1 - m_1 F_2 \right) + \left(m_2 \vec{F_{12}} - m_1 \vec{F_{21}} \right)$$

$$= \left(m_2 \vec{F_1} - m_1 \vec{F_2} \right) + \left[m_2 \vec{F_{12}} - m_1 \left(-\vec{F_{12}} \right) \right]$$

$$\left[\because \vec{F_{21}} = -\vec{F_{12}} \right]$$

$$= \left(m_2 \vec{F_1} - m_1 \vec{F_2} \right) + \left[m_2 \vec{F_{12}} + m_1 \vec{F_{12}} \right]$$

$$= \left(m_2 \vec{F_1} - m_1 \vec{F_2} \right) + \left(m_1 + m_2 \right) \vec{F_{12}}$$

$$= m_1 m_2 \left(\frac{\vec{F_1}}{m_1} - \frac{\vec{F_2}}{m_2} \right) + \left(m_1 + m_2 \right) \vec{F_{12}} \quad \text{--- (28)}$$

eqn (28) $\longrightarrow$

$$m_1 m_2 \frac{d^2}{dt^2}(\vec{r_1} - \vec{r_2}) = m_1 m_2 \left(\frac{\vec{F_1}}{m_1} - \frac{\vec{F_2}}{m_2}\right) + (m_1 + m_2)\vec{F_{12}}$$

• eqn (28) $\div (m_1 + m_2) \Rightarrow$

$$\frac{m_1 m_2}{(m_1 + m_2)} \frac{d^2}{dt^2}(\vec{r_1} - \vec{r_2}) = \frac{m_1 m_2}{(m_1 + m_2)} \left(\frac{\vec{F_1}}{m_1} - \frac{\vec{F_2}}{m_2}\right) + \vec{F_{12}}$$

or, $$\boxed{\mu \frac{d^2}{dt^2}(\vec{r_1} - \vec{r_2}) = \mu \left(\frac{\vec{F_1}}{m_1} - \frac{\vec{F_2}}{m_2}\right) + \vec{F_{12}}} \rightarrow (29)$$

Where $\mu = \frac{m_1 m_2}{m_1 + m_2}$

= reduced mass considering 2-body system

Case I

If the motion of the two-body system is only due to the external forces arising due to the interaction between the two particles, then the eqn of motion of (29) will be reduced to

$$\mu \frac{d^2(\vec{r}_1 - \vec{r}_2)}{dt^2} = \vec{F}_{12} \rightarrow (30),$$

Since the term $\left(\frac{\vec{F}_1}{m_1} - \frac{\vec{F}_2}{m_2}\right) = 0$.

Case 2

If $\vec{r} = \vec{r}_1 - \vec{r}_2$

= relative separatn betn m_1 & m_2,

then eqn (30) can be written as,

$$\frac{d^2\vec{r}}{dt^2} = \vec{F}_{12} \quad \text{—(31)}.$$

eqn (31) denotes the equation of motion of a body of mass μ acted on by the internal for $\vec{F}_{12}$ which the body of mass m_2 exerts on the body of mass m_1.

- Thus the motion of m_1 when viewed from m_2 is the same as if m_2 is fixed and the body in motion has a mass μ.
- In this way, two-body-problem can be considered as a single body problem by assuming that one body is fixed and another in motion having reduced mass.