University of Calcutta

Semester 2

PHYSICS

Paper: PHS-G-CC-2-2-TH (NEW SYLLABUS)

VECTOR INTEGRATION, GAUSS'S DIVERGENCE THEOREM , STOKES THEOREM

SOLVED PROBLEMS

ASSIGNMENTS

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VECTOR INTEGRATION

Introduction of Line Integral

A line integral (sometimes called a path integral) is the integral of some function along a curve. One can integrate a scalar-valued function along a curve, obtaining for example, the mass of a wire from its density. One can also integrate a certain type of vector-valued functions along a curve. These vector-valued functions are the ones where the input and output dimensions are the same, and we usually represent them as vector fields.

Line integral

LINE INTEGRALS. Let $\mathbf{r}(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$, where $\mathbf{r}(u)$ is the position vector of (x,y,z), define a curve C joining points P_1 and P_2, where $u = u_1$ and $u = u_2$ respectively.

We assume that C is composed of a finite number of curves for each of which $\mathbf{r}(u)$ has a continuous derivative. Let $\mathbf{A}(x,y,z) = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of $\mathbf{A}$ along C from P_1 to P_2, written as

$$\int_{P_1}^{P_2} \mathbf{A} \cdot d\mathbf{r} = \int_C \mathbf{A} \cdot d\mathbf{r} = \int_C A_1\,dx + A_2\,dy + A_3\,dz$$

is an example of a *line integral*. If $\mathbf{A}$ is the force $\mathbf{F}$ on a particle moving along C, this line integral represents the work done by the force. If C is a closed curve (which we shall suppose is a *simple closed curve*, i.e. a curve which does not intersect itself anywhere) the integral around C is often denoted by

$$\oint \mathbf{A} \cdot d\mathbf{r} = \oint A_1\,dx + A_2\,dy + A_3\,dz$$

Clear your idea

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Surface Integral

SURFACE INTEGRALS. Let S be a two-sided surface, such as shown in the figure below. Let one side of S be considered arbitrarily as the positive side (if S is a closed surface this is taken as the outer side). A unit normal $\mathbf{n}$ to any point of the positive side of S is called a *positive* or *outward drawn* unit normal.

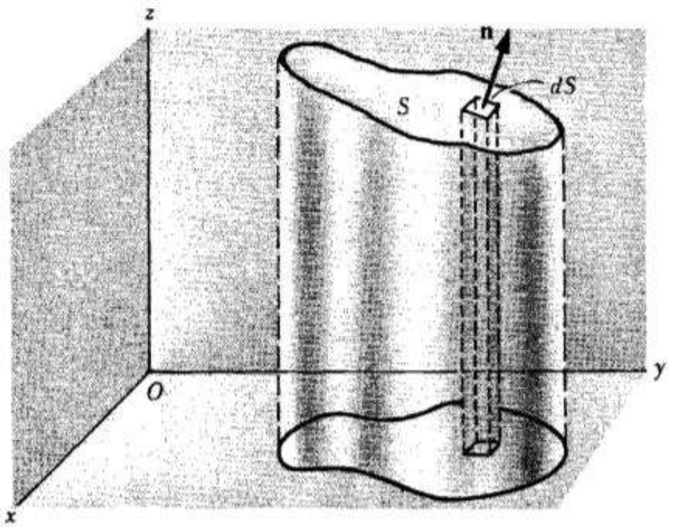
Associate with the differential of surface area dS a vector $d\mathbf{S}$ whose magnitude is dS and whose direction is that of $\mathbf{n}$. Then $d\mathbf{S} = \mathbf{n}\, dS$. The integral

$$\iint_S \mathbf{A}\cdot d\mathbf{S} = \iint_S \mathbf{A}\cdot\mathbf{n}\, dS$$

is an example of a surface integral called the *flux* of $\mathbf{A}$ over S. Other surface integrals are

$$\iint_S \phi\, dS, \quad \iint_S \phi\, \mathbf{n}\, dS, \quad \iint_S \mathbf{A}\times d\mathbf{S}$$

where ϕ is a scalar function. Such integrals can be defined in terms of limits of sums as in elementary calculus (see Problem 17).



The notation $\oiint_S$ is sometimes used to indicate integration over the closed surface S. Where no confusion can arise the notation $\oint_S$ may also be used.

To evaluate surface integrals, it is convenient to express them as double integrals taken over the projected area of the surface S on one of the coordinate planes. This is possible if any line perpendicular to the coordinate plane chosen meets the surface in no more than one point. However, this does not pose any real problem since we can generally subdivide S into surfaces which do satisfy this restriction.

The line integral of a vector field $\mathbf{F}$ could be interpreted as the work done by the force field $\mathbf{F}$ on a particle moving along the path. The surface integral of a vector field $\mathbf{F}$ actually has a simpler explanation. If the vector field $\mathbf{F}$ represents the flow of a fluid, then the surface integral of $\mathbf{F}$ will represent the amount of fluid flowing through the surface (per unit time).

The amount of the fluid flowing through the surface per unit time is also called the **flux** of fluid through the surface. For this reason, we often call the surface integral of a vector field a **flux integral**.

Clear your idea

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Volume Integral

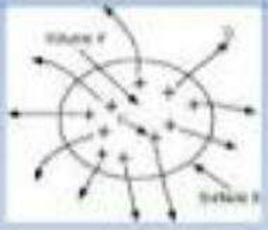
VOLUME INTEGRALS. Consider a closed surface in space enclosing a volume V. Then

$$\iiint_V \mathbf{A}\, dV \quad \text{and} \quad \iiint_V \phi\, dV$$

are examples of *volume integrals* or *space integrals* as they are sometimes called. For evaluation of such integrals, see the Solved Problems.

Comparison between surface Volume integral

Surface & Volume Integral

	Surface integral	Volume integral
Diagram		
Maths description	$\Psi = \int_S \mathbf{A} \cdot d\mathbf{S}$	$\int_v \rho_v \, dv$
Result	A measure of the total flux from **vector** field passing through a given surface	A measure of the total effect of a scalar **function** i.e. temperature, inside a given volume
Information required	1. Vector field expression ***A*** 2. Surface expression	1. Scalar Function ρ_v 2. Volume expression

Integral limits depends on surface

Integral limits depends on volume

The Gauss Theorem

This theorem is related to conservation laws in physics. It states that the total sources and sinks of a vectorial quantity, or the integral volume of its divergence, is equal to the net flux of this vectorial quantity across the volume boundary.

$$\int_{\Delta V} d^3x \nabla \cdot \mathbf{A} = \oint_{\Delta S} \mathbf{A} \cdot d\mathbf{S}$$

Gauss's Divergence theorem

$$\iint_S \vec{A}\cdot \hat{n}\,dS = \iiint_V (\text{div } \vec{A})\,dV.$$

This is Gauss's divergence theorem, i.e., the volume integral of the divergence of a vector $\vec{A}$ in a vector field, taken through a given volume, is equal to the surface integral of the normal component of $\vec{A}$ over the surface which encloses the volume.

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STOKE'S THEOREM

The circulation of a vector field **A** around a closed path L is equal to the surface integral of the curl of **A** over the open surface S bounded by L that **A** and curl of **A** are continuous on S.

$$\oint_L \mathbf{A} \bullet dl = \int_S (\nabla \times \mathbf{A}) \bullet dS$$

Stokes Theorem

$$\therefore \oint_L \vec{A} \cdot d\vec{l} = \iint_S \text{curl}\, \vec{A} \cdot d\vec{S}.$$

This is Stokes's theorem. It can be expressed in the following way:

The line integral of the tangential component of the vector $\vec{A}$ *round any closed path is equal to the normal surface integral of the vector* curl $\vec{A}$ *over the surface having the path as its boundary*

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Clear your idea

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Solved Problem

1. If $\mathbf{R}(u) = (u - u^2)\mathbf{i} + 2u^3\,\mathbf{j} - 3\mathbf{k}$, find (*a*) $\int \mathbf{R}(u)\,du$ and (*b*) $\int_1^2 \mathbf{R}(u)\,du$.

(*a*)
$$\begin{aligned}
\int \mathbf{R}(u)\,du &= \int \left[(u-u^2)\mathbf{i} + 2u^3\,\mathbf{j} - 3\mathbf{k}\right] du \\
&= \mathbf{i}\int (u-u^2)\,du + \mathbf{j}\int 2u^3\,du + \mathbf{k}\int -3\,du \\
&= \mathbf{i}\left(\frac{u^2}{2} - \frac{u^3}{3} + c_1\right) + \mathbf{j}\left(\frac{u^4}{2} + c_2\right) + \mathbf{k}(-3u + c_3) \\
&= \left(\frac{u^2}{2} - \frac{u^3}{3}\right)\mathbf{i} + \frac{u^4}{2}\mathbf{j} - 3u\,\mathbf{k} + c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k} \\
&= \left(\frac{u^2}{2} - \frac{u^3}{3}\right)\mathbf{i} + \frac{u^4}{2}\mathbf{j} - 3u\,\mathbf{k} + \mathbf{c}
\end{aligned}$$

where $\mathbf{c}$ is the constant vector $c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$.

(*b*) From (*a*),
$$\begin{aligned}
\int_1^2 \mathbf{R}(u)\,du &= \left(\frac{u^2}{2} - \frac{u^3}{3}\right)\mathbf{i} + \frac{u^4}{2}\mathbf{j} - 3u\,\mathbf{k} + \mathbf{c}\,\Big|_1^2 \\
&= \left[\left(\frac{2^2}{2} - \frac{2^3}{3}\right)\mathbf{i} + \frac{2^4}{2}\mathbf{j} - 3(2)\mathbf{k} + \mathbf{c}\right] - \left[\left(\frac{1^2}{2} - \frac{1^3}{3}\right)\mathbf{i} + \frac{1^4}{2}\mathbf{j} - 3(1)\mathbf{k} + \right. \\
&= -\frac{5}{6}\mathbf{i} + \frac{15}{2}\mathbf{j} - 3\mathbf{k}
\end{aligned}$$

Another Method.

$$\begin{aligned}
\int_1^2 \mathbf{R}(u)\,du &= \mathbf{i}\int_1^2 (u-u^2)\,du + \mathbf{j}\int_1^2 2u^3\,du + \mathbf{k}\int_1^2 -3\,du \\
&= \mathbf{i}\left(\frac{u^2}{2} - \frac{u^3}{3}\right)\Big|_1^2 + \mathbf{j}\left(\frac{u^4}{2}\right)\Big|_1^2 + \mathbf{k}(-3u)\Big|_1^2 = -\frac{5}{6}\mathbf{i} + \frac{15}{2}\mathbf{j} - 3\mathbf{k}
\end{aligned}$$

3. Evaluate $\int \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2}\, dt$.

$$\frac{d}{dt}\left(\mathbf{A} \times \frac{d\mathbf{A}}{dt}\right) = \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2} + \frac{d\mathbf{A}}{dt} \times \frac{d\mathbf{A}}{dt} = \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2}$$

Integrating, $\int \mathbf{A} \times \frac{d^2\mathbf{A}}{dt^2}\, dt = \int \frac{d}{dt}\left(\mathbf{A} \times \frac{d\mathbf{A}}{dt}\right) dt = \mathbf{A} \times \frac{d\mathbf{A}}{dt} + \mathbf{c}$.

6. If $\mathbf{A} = (3x^2 + 6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\,\mathbf{k}$, evaluate $\int_C \mathbf{A} \cdot d\mathbf{r}$ from (0,0,0) to (1,1,1) along the following paths C:

(*a*) $x = t,\ y = t^2,\ z = t^3$.

(*b*) the straight lines from (0,0,0) to (1,0,0), then to (1,1,0), and then to (1,1,1).

(*c*) the straight line joining (0,0,0) and (1,1,1).

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \int_C \left[(3x^2+6y)\mathbf{i} - 14yz\,\mathbf{j} + 20xz^2\mathbf{k}\right] \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

(*a*) If $x = t,\ y = t^2,\ z = t^3$, points (0,0,0) and (1,1,1) correspond to $t = 0$ and $t = 1$ respectively. Then

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \int_{t=0}^{1} (3t^2 + 6t^2)\,dt - 14(t^2)(t^3)\,d(t^2) + 20(t)(t^3)^2\,d(t^3)$$

$$= \int_{t=0}^{1} 9t^2\,dt - 28t^6\,dt + 60t^9\,dt$$

$$= \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\,dt = 3t^3 - 4t^7 + 6t^{10}\Big|_0^1 = 5$$

Another Method.

Along C, $\mathbf{A} = 9t^2\mathbf{i} - 14t^5\mathbf{j} + 20t^7\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ and $d\mathbf{r} = (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k})\,dt$.

Then $\int_C \mathbf{A} \cdot d\mathbf{r} = \int_{t=0}^{1} (9t^2\,\mathbf{i} - 14t^5\,\mathbf{j} + 20t^7\,\mathbf{k}) \cdot (\mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k})\,dt$

$$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\,dt = 5$$

(*b*) Along the straight line from (0,0,0) to (1,0,0) $y = 0,\ z = 0,\ dy = 0,\ dz = 0$ while x varies from 0 to 1. Then the integral over this part of the path is

$$\int_{x=0}^{1} \left(3x^2 + 6(0)\right) dx - 14(0)(0)(0) + 20x(0)^2(0) = \int_{x=0}^{1} 3x^2\,dx = x^3\Big|_0^1 = 1$$

Along the straight line from (1,0,0) to (1,1,0) $x = 1,\ z = 0,\ dx = 0,\ dz = 0$ while y varies from 0 to 1. Then the integral over this part of the path is

$$\int_{y=0}^{1} (3(1)^2 + 6y)0 - 14y(0)\,dy + 20(1)(0)^2\,0 = 0$$

Along the straight line from (1,1,0) to (1,1,1) $x = 1,\ y = 1,\ dx = 0,\ dy = 0$ while z varies from 0 to 1. Then the integral over this part of the path is

$$\int_{z=0}^{1} \left(3(1)^2 + 6(1)\right)0 - 14(1)\,z(0) + 20(1)\,z^2\,dz = \int_{z=0}^{1} 20z^2\,dz = \frac{20z^3}{3}\Big|_0^1 = \frac{20}{3}$$

Adding, $\int_C \mathbf{A} \cdot d\mathbf{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$

(*c*) The straight line joining (0,0,0) and (1,1,1) is given in parametric form by $x = t,\ y = t,\ z = t$. Then

$$\int_C \mathbf{A} \cdot d\mathbf{r} = \int_{t=0}^{1} (3t^2 + 6t)\,dt - 14(t)(t)\,dt + 20(t)(t)^2\,dt$$

$$= \int_{t=0}^{1} (3t^2 + 6t - 14t^2 + 20t^3)\,dt = \int_{t=0}^{1} (6t - 11t^2 + 20t^3)\,dt = \frac{13}{3}$$

7. Find the total work done in moving a particle in a force field given by $\mathbf{F} = 3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=1$ to $t=2$.

$$\text{Total work} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C 3xy\,dx - 5z\,dy + 10x\,dz$$

$$= \int_{t=1}^{2} 3(t^2+1)(2t^2)\,d(t^2+1) - 5(t^3)\,d(2t^2) + 10(t^2+1)\,d(t^3)$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt = 303$$

11. (*a*) If $\mathbf{F}$ is a conservative field, prove that $\text{curl}\,\mathbf{F} = \nabla\times\mathbf{F} = \mathbf{0}$ (i.e. $\mathbf{F}$ is irrotational).
(*b*) Conversely, if $\nabla\times\mathbf{F} = \mathbf{0}$ (i.e. $\mathbf{F}$ is irrotational), prove that $\mathbf{F}$ is conservative.

(*a*) If $\mathbf{F}$ is a conservative field, then by Problem 10, $\mathbf{F} = \nabla\phi$.
Thus $\text{curl}\,\mathbf{F} = \nabla\times\nabla\phi = \mathbf{0}$ (see Problem 27(*a*), Chapter 4).

(*b*) If $\nabla\times\mathbf{F} = \mathbf{0}$, then
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \mathbf{0} \quad \text{and thus}$$

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

We must prove that $\mathbf{F} = \nabla\phi$ follows as a consequence of this.

The work done in moving a particle from (x_1,y_1,z_1) to (x,y,z) in the force field $\mathbf{F}$ is

$$\int_C F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz$$

where C is a path joining (x_1,y_1,z_1) and (x,y,z). Let us choose as a particular path the straight line segments from (x_1,y_1,z_1) to (x,y_1,z_1) to (x,y,z_1) to (x,y,z) and call $\phi(x,y,z)$ the work done along this particular path. Then

$$\phi(x,y,z) = \int_{x_1}^{x} F_1(x,y_1,z_1)\,dx + \int_{y_1}^{y} F_2(x,y,z_1)\,dy + \int_{z_1}^{z} F_3(x,y,z)\,dz$$

It follows that

$$\frac{\partial\phi}{\partial z} = F_3(x,y,z)$$

$$\frac{\partial\phi}{\partial y} = F_2(x,y,z_1) + \int_{z_1}^{z} \frac{\partial F_3}{\partial y}(x,y,z)\,dz$$

$$= F_2(x,y,z_1) + \int_{z_1}^{z} \frac{\partial F_2}{\partial z}(x,y,z)\,dz$$

$$= F_2(x,y,z_1) + F_2(x,y,z)\Big|_{z_1}^{z} = F_2(x,y,z_1) + F_2(x,y,z) - F_2(x,y,z_1) = F_2(x,y,z)$$

$$\frac{\partial\phi}{\partial x} = F_1(x,y_1,z_1) + \int_{y_1}^{y} \frac{\partial F_2}{\partial x}(x,y,z_1)\,dy + \int_{z_1}^{z} \frac{\partial F_3}{\partial x}(x,y,z)\,dz$$

$$= F_1(x,y_1,z_1) + \int_{y_1}^{y} \frac{\partial F_1}{\partial y}(x,y,z_1)\,dy + \int_{z_1}^{z} \frac{\partial F_1}{\partial z}(x,y,z)\,dz$$

$$= F_1(x,y_1,z_1) + F_1(x,y,z_1)\Big|_{y_1}^{y} + F_1(x,y,z)\Big|_{z_1}^{z}$$

$$= F_1(x,y_1,z_1) + F_1(x,y,z_1) - F_1(x,y_1,z_1) + F_1(x,y,z) - F_1(x,y,z_1) = F_1(x,y,z)$$

Then
$$\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} = \nabla\phi.$$

Thus a necessary and sufficient condition that a field $\mathbf{F}$ be conservative is that $\text{curl}\,\mathbf{F} = \nabla\times\mathbf{F} = \mathbf{0}$.

21. Evaluate $\iint\limits_S \phi \mathbf{n}\, dS$ where $\phi = \frac{3}{8}xyz$ and S is the surface of Problem 20.

We have
$$\iint\limits_S \phi \mathbf{n}\, dS = \iint\limits_R \phi \mathbf{n} \frac{dx\, dz}{|\mathbf{n}\cdot\mathbf{j}|}$$

Using $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j}}{4}$, $\mathbf{n}\cdot\mathbf{j} = \frac{y}{4}$ as in Problem 20, this last integral becomes

$$\iint\limits_R \frac{3}{8}xz\,(x\mathbf{i} + y\mathbf{j})\, dx\, dz = \frac{3}{8}\int_{z=0}^{5}\int_{x=0}^{4} (x^2 z\,\mathbf{i} + xz\sqrt{16-x^2}\,\mathbf{j})\, dx\, dz$$

$$= \frac{3}{8}\int_{z=0}^{5} \left(\frac{64}{3}z\,\mathbf{i} + \frac{64}{3}z\,\mathbf{j}\right) dz = 100\,\mathbf{i} + 100\,\mathbf{j}$$

28. A fluid of density $\rho(x,y,z,t)$ moves with velocity $\mathbf{v}(x,y,z,t)$. If there are no sources or sinks, prove that

$$\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0 \qquad \text{where } \mathbf{J} = \rho\mathbf{v}$$

Consider an arbitrary surface enclosing a volume V of the fluid. At any time the mass of fluid within V is

$$M = \iiint_V \rho\, dV$$

The time rate of increase of this mass is

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\iiint_V \rho\, dV = \iiint_V \frac{\partial\rho}{\partial t}\, dV$$

The mass of fluid per unit time leaving V is

$$\iint_S \rho\mathbf{v}\cdot\mathbf{n}\, dS$$

(see Problem 15) and the time rate of increase in mass is therefore

$$-\iint_S \rho\mathbf{v}\cdot\mathbf{n}\, dS = -\iiint_V \nabla\cdot(\rho\mathbf{v})\, dV$$

by the divergence theorem. Then

$$\iiint_V \frac{\partial\rho}{\partial t}\, dV = -\iiint_V \nabla\cdot(\rho\mathbf{v})\, dV$$

or

$$\iiint_V (\nabla\cdot(\rho\mathbf{v}) + \frac{\partial\rho}{\partial t})\, dV = 0$$

Since V is arbitrary, the integrand, assumed continuous, must be identically zero, by reasoning similar to that used in Problem 12. Then

$$\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0 \qquad \text{where } \mathbf{J} = \rho\mathbf{v}$$

The equation is called the *continuity equation*. If ρ is a constant, the fluid is incompressible and $\nabla\cdot\mathbf{v} = 0$, i.e. $\mathbf{v}$ is solenoidal.

The continuity equation also arises in electromagnetic theory, where ρ is the *charge density* and $\mathbf{J} = \rho\mathbf{v}$ is the *current density*.

32. Verify Stokes' theorem for $\mathbf{A} = (2x-y)\mathbf{i} - yz^2\mathbf{j} - y^2z\mathbf{k}$, where S is the upper half surface of the sphere $x^2+y^2+z^2 = 1$ and C is its boundary.

The boundary C of S is a circle in the xy plane of radius one and center at the origin. Let $x = \cos t$, $y = \sin t$, $z = 0$, $0 \leqq t < 2\pi$ be parametric equations of C. Then

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \oint_C (2x-y)\,dx - yz^2\,dy - y^2z\,dz$$

$$= \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)\,dt = \pi$$

Also,
$$\nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \mathbf{k}$$

Then
$$\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS = \iint_S \mathbf{k}\cdot\mathbf{n}\,dS = \iint_R dx\,dy$$

since $\mathbf{n}\cdot\mathbf{k}\,dS = dx\,dy$ and R is the projection of S on the xy plane. This last integral equals

$$\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1\int_0^{\sqrt{1-x^2}} dy\,dx = 4\int_0^1 \sqrt{1-x^2}\,dx = \pi$$

and Stokes' theorem is verified.

Assignments

(d) Find the line integral of the vector $\vec{F}$ along the line segment joining the points (0, 0, 0) and (2, 3, 4) — $\vec{F} = x^2\hat{\imath} + y^2\hat{\jmath} + (xz - y)\hat{k}$. [C.U. (Hons.) 1999].

30. (a) Prove the identity $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 A$. [C.U. (Hons.) 2000]

(b) State Stokes' theorem. Using this theorem prove that

$$\oint_C \phi d\vec{r} = \iint_S d\vec{S} \times \vec{\nabla}\phi,$$

where ϕ is a scalar function of $\vec{r}$, S is an open surface bounded by a closed curve. [C.U. (Hons.) 1993, 2000]

Answer of (b) We consider $\vec{F} = \vec{a}\phi$, where $\vec{a}$ is a constant vector and ϕ is a differentiable scalar point function.

Then

$$\int_C (\vec{a}\phi) \cdot d\vec{r} = \iint_S \text{curl}(\vec{a}\phi) \cdot \hat{n} dS \text{ (using Stokes' theorem)}$$

$$= \iint_S (\phi \,\text{curl}\, \vec{a} - \vec{a} \times \text{grad}\, \phi) \cdot \hat{n} dS = \iint_S (\vec{\nabla}\phi \times \vec{a}) \cdot \hat{n} dS$$

$$= -\iint_S \vec{a} \cdot (\vec{\nabla}\phi \times \hat{n}) dS = -\vec{a} \cdot \iint_S (\vec{\nabla}\phi \times \hat{n}) dS$$

$$\text{or} \quad \vec{a} \cdot \int_C \phi d\vec{r} = \vec{a} \cdot \iint_S (\hat{n} \times \vec{\nabla}\phi) dS$$

$$\text{or} \quad \int_C \phi dr = \iint_S d\vec{S} \times \vec{\nabla}\phi \quad \text{(since } \vec{a} \text{ is arbitrary)}$$

31. Prove the identity $(\vec{A} \cdot \vec{\nabla})\vec{A} = \frac{1}{2}\vec{\nabla}A^2 - \vec{A} \times (\vec{\nabla} \times \vec{A})$. [C.U. (Hons.) 1993]

[Ans. See worked out examples.]

32. Verify divergence theorem for a vector $\vec{A} = \hat{\imath}\frac{x}{r} + \hat{\jmath}\frac{y}{r} + \hat{k}\frac{z}{r} = \frac{\vec{r}}{r}$, where $\hat{\imath}, \hat{\jmath}, \hat{k}$ are the unit vectors along X, Y, Z — the region of integration being a sphere of radius a with centre at the origin.

[C.U. (Hons.) 1980]

33. Show that each of the three equations

$$\vec{F} = \vec{\nabla}\phi, \quad \vec{\nabla} \times \vec{F} = 0, \quad \oint \vec{F} \cdot d\vec{r} = 0,$$

implies the other two.

34. Prove that when $\vec{B}$ is solenoidal, a vector $\vec{A}$ exists such that $\vec{B} = \vec{\nabla} \times \vec{A}$.

35. If $\vec{F} = 3xy\hat{\imath} - y^2\hat{\jmath}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the curve in the $x-y$ plane, $y = 2x^2$, from (0, 0) to (1, 2). [Ans. $-\frac{7}{6}$.]