University of Calcutta
Semester 2
PHYSICS
Paper: PHS-G-CC-2-2-TH (NEW SYLLABUS)

VECTOR ALGEBRA- Part 1
SOLVED PROBLEMS
ASSIGNMENTS

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Introduction of VECTOR ALBRA

Q1. What is a Vector in Math?

A1. We can define a vector as an object that has both a direction and a magnitude. Geometrically, we can represent a vector as a directed line segment, whose length is the magnitude of the vector and with an arrow indicating the direction. Moreover, two examples of vectors are those that characterize force and velocity.

Q2. How do vectors work?

A2. Vectors refers to lines that represent both direction and magnitude (size). Consequently, if an object move in more than one direction or more than one force acts upon an object equivalently, then we can add the vectors to find a resultant displacement or resultant force that acts on it.

Q3. What does || A || mean?

A3. "‖" means that the lines are parallel to each other. In a square, all the opposite sides are parallel which means that they never intersect each other at any point and are parallel to infinity. The most practical example of parallel lines is railway tracks that never join each other.

Q4. Is position a vector?

A4. Yes, the position is a vector quantity. Because it has a magnitude (size) as well as a direction. Moreover, the magnitude of a vector quantity is a number (with units) that tells how much of the quantity there is and the direction tells you which way the vector is pointing.

Basic concept about Vector

https://youtu.be/Y0eng_h4TKU

https://youtu.be/_YkIivLaVJs

Acknowledgement

1. www.topper.com

2. A treatise on general properties of matter : Sengupta and Chatterjee

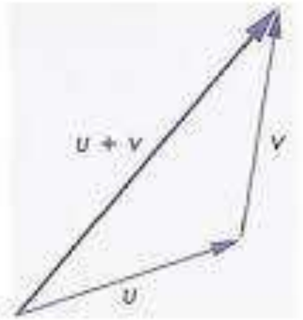
Basic rules :Addition and Multiplication of a vector quantity with a scalar quantity

In handling vector quantities the following laws of Vector Algebra are to be noted.

1. $\vec{A} + \vec{B} = \vec{B} + \vec{A}$—Commutative law for addition.
2. $K\vec{A} = \vec{A}K$—Commutative law for multiplication.
3. $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$—Associative law for addition.
4. $K(l\vec{A}) = (Kl)\vec{A}$—Associative law for multiplication.
5. $\vec{A}(K + l) = K\vec{A} + l\vec{A}$—Distributive law.
6. $K(\vec{A} + \vec{B}) = K\vec{A} + K\vec{B}$—Distributive law.

Vector operations – intuitively

- Vector-Vector Addition
  - Visualize using <u>head-to-tail axiom</u>



Head-to-tail axiom

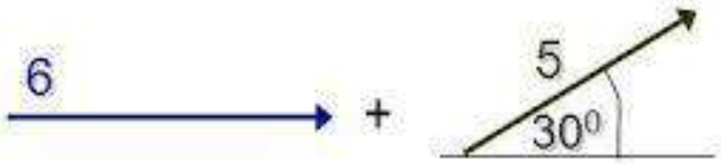
- Scalar-vector multiplication
  - Resize (scale) the directed line segment

α > 1	length increases
0 > α > 1	length decreases
α < 0	reverse direction

Scalar-vector multi.

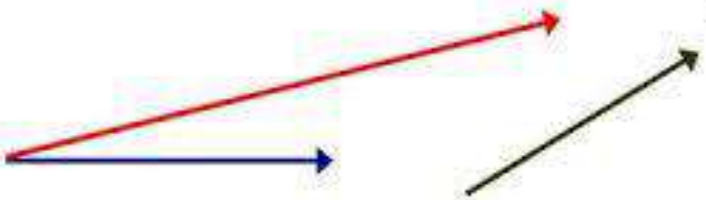
Vector addition - head-to-tail method

vectors: 6 units,E + 5 units,30^0



examples:
$\vec{v}$ – velocity: 6 m/s, E + 5 m/s, 30^0
$\vec{a}$ – acceleration: 6 m/s^2, E + 5 m/s^2, 30^0
$\vec{F}$ – force: 6 N, E + 5 N, 30^0

you can ONLY add the same kind (apples + apples)

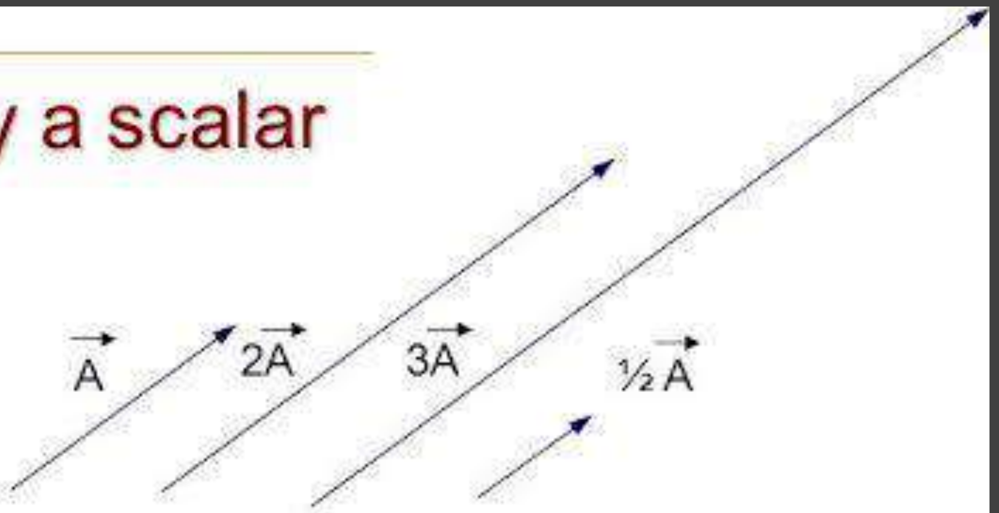


1. Vectors are drawn to scale in given direction.
2. The second vector is then drawn such that its tail is positioned at the head of the first vector.
3. The sum of two such vectors is the third vector which stretches from the tail of the first vector to the head of the second vector.

This third vector is known as the "resultant" - it is the result of adding the two vectors. The resultant is the vector sum of the two individual vectors. So, you can see now that magnitude of the resultant is dependent upon the direction which the two individual vectors have.

Multiplying vector by a scalar

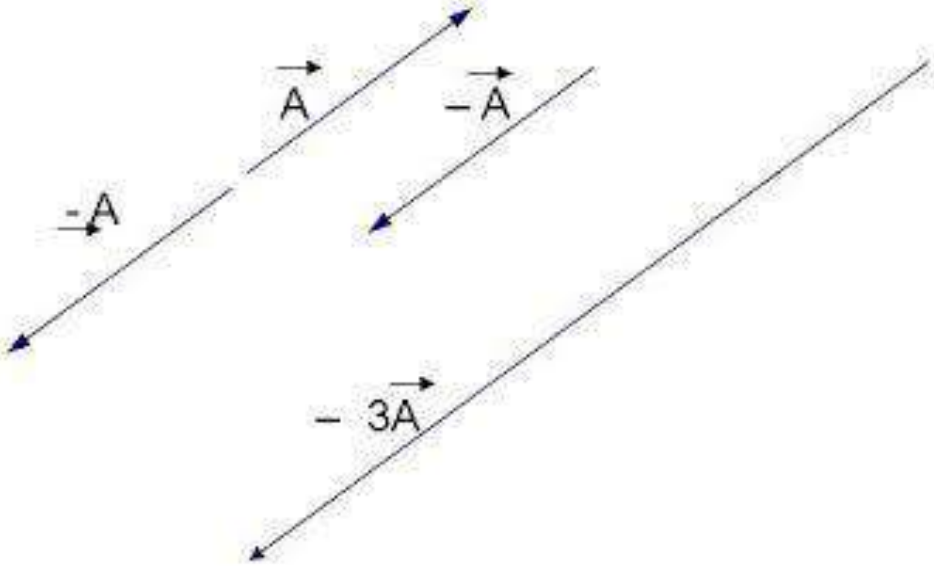
Multiplying a vector by a scalar will ONLY CHANGE its **magnitude** – not direction.



Opposite vectors

One exception:

Multiplying a vector by "-1" does not change the magnitude, but it does reverse it's direction



Clear your idea more

https://youtu.be/dDWnH6mwyS8

VECTOR PRODUCT

When two vectors $\bar{A}$ and $\bar{B}$ are multiplied, the result is either a scalar or a vector depending on how they are multiplied. There are two types of vector multiplication:

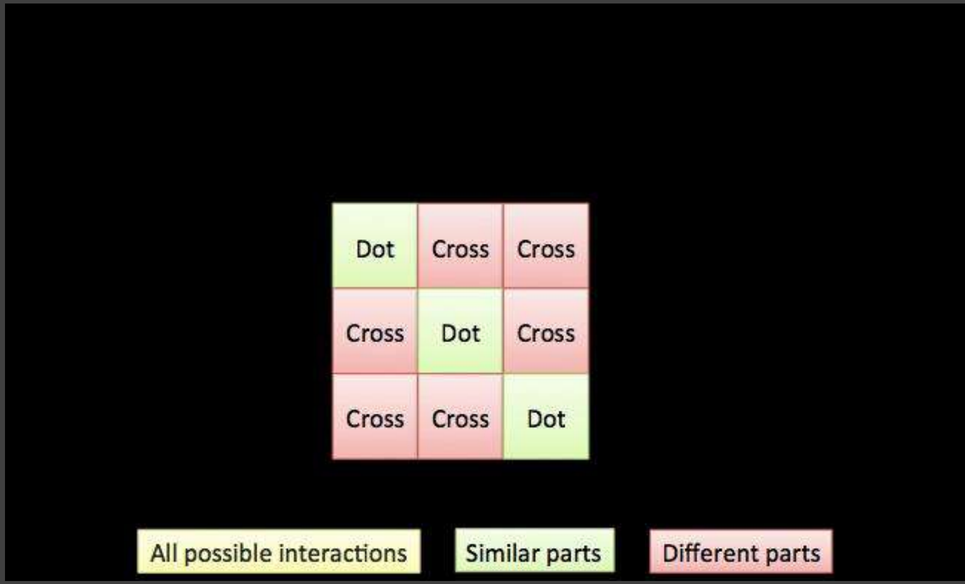
1. Scalar (or dot) product: $\bar{A} \cdot \bar{B}$

2. Vector (or cross) product: $\bar{A} \times \bar{B}$

The dot product of the two vectors $\bar{A}$ and $\bar{B}$ is defined geometrically as the product of the magnitude of $\bar{B}$ and the projection of $\bar{A}$ onto $\bar{B}$ (or vice versa):

$$\bar{A} \cdot \bar{B} = AB\cos\theta_{AB}$$

where θ_{AB} is the smaller angle between $\bar{A}$ and $\bar{B}$



Interesting diagram

DOT PRODUCT

Scalar Product

$$\vec{A} \cdot \vec{B} = |A||B|cos\emptyset$$

Or

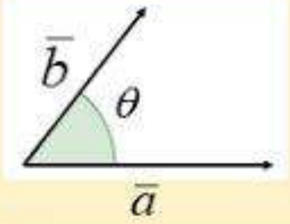
$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y$$

Dot Product

If $\bar{a} = < a_1, a_2, a_3 >$ and $\bar{b} = < b_1, b_2, b_3 >$

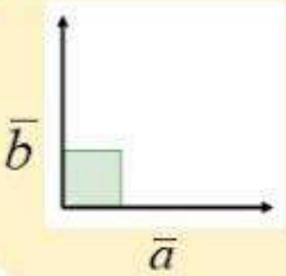
then the dot product is

$$\bar{a} \bullet \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$



If θ is the angle between $\bar{a}$ and $\bar{b}$ then

$$\bar{a} \bullet \bar{b} = |\bar{a}||\bar{b}|\cos\theta$$



$\bar{a} \bullet \bar{b}$ are orthogonal (perpendicular)

if and only if $\bar{a} \bullet \bar{b} = 0$

EXAMPLES OF DOT PRODUCT

Dot product example

- Determine the angle between the following two vectors:

$$\vec{A} = -7\hat{i} + 4\hat{j}$$

$$\vec{B} = -2\hat{i} + 9\hat{j}$$

$$A = \sqrt{65}$$

$$B = \sqrt{85}$$

$$\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y$$

$$(-7)(-2) + (4)(9) = 50$$

$$50 = \sqrt{(65)(85)} \cos\theta$$

$$\theta = \cos^{-1} \frac{50}{\sqrt{(65)(85)}}$$

$$\theta = 47.7^0$$

Example:

$$\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k} \qquad |\mathbf{A}| = \sqrt{3}$$

$$\mathbf{B} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k} \qquad |\mathbf{B}| = \sqrt{6}$$

$$\mathbf{A} \cdot \mathbf{B} = -1 + 2 - 1 = 0$$

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{0}{\sqrt{18}} = 0$$

$$\theta = \pi/2$$

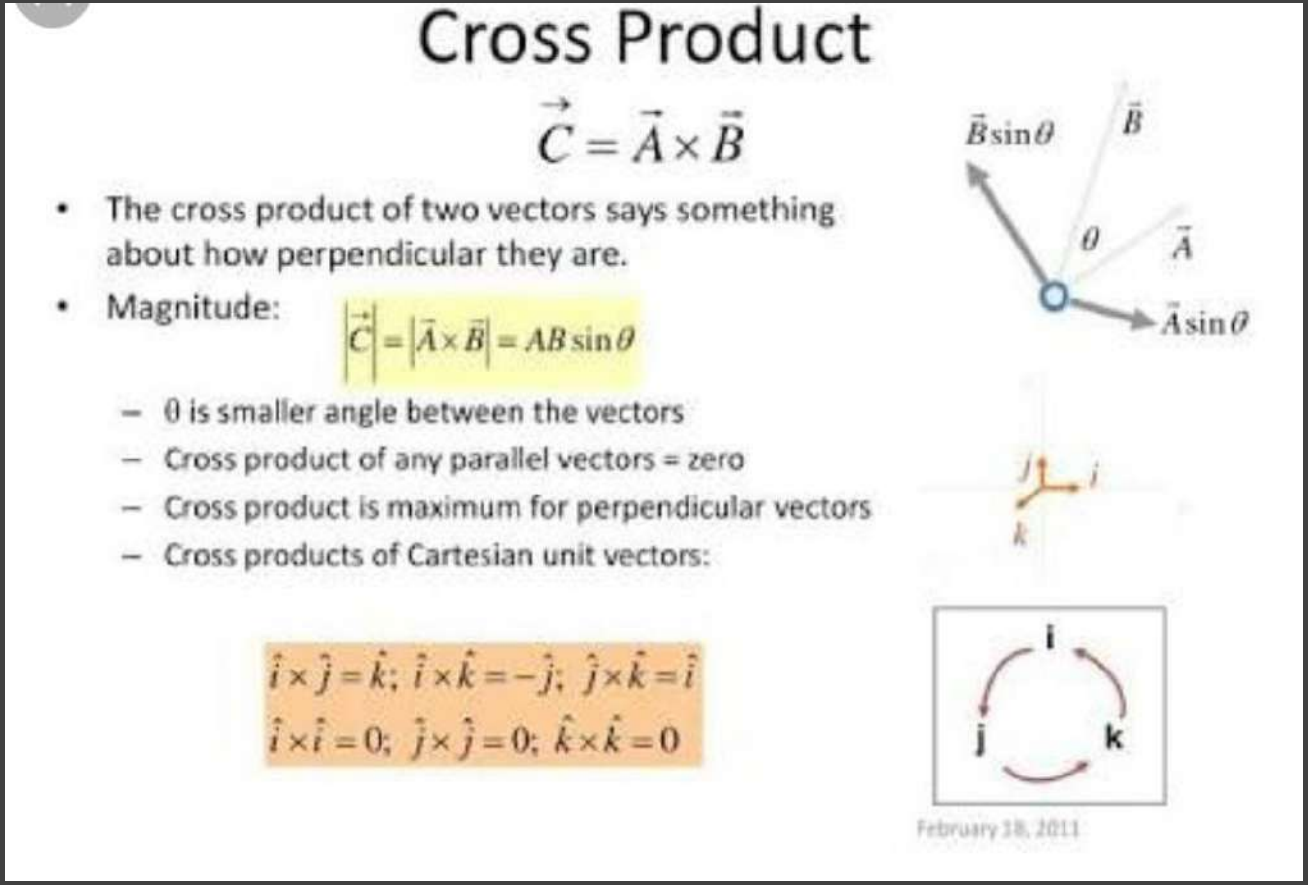
$\perp$ Vectors have a dot product of 0.

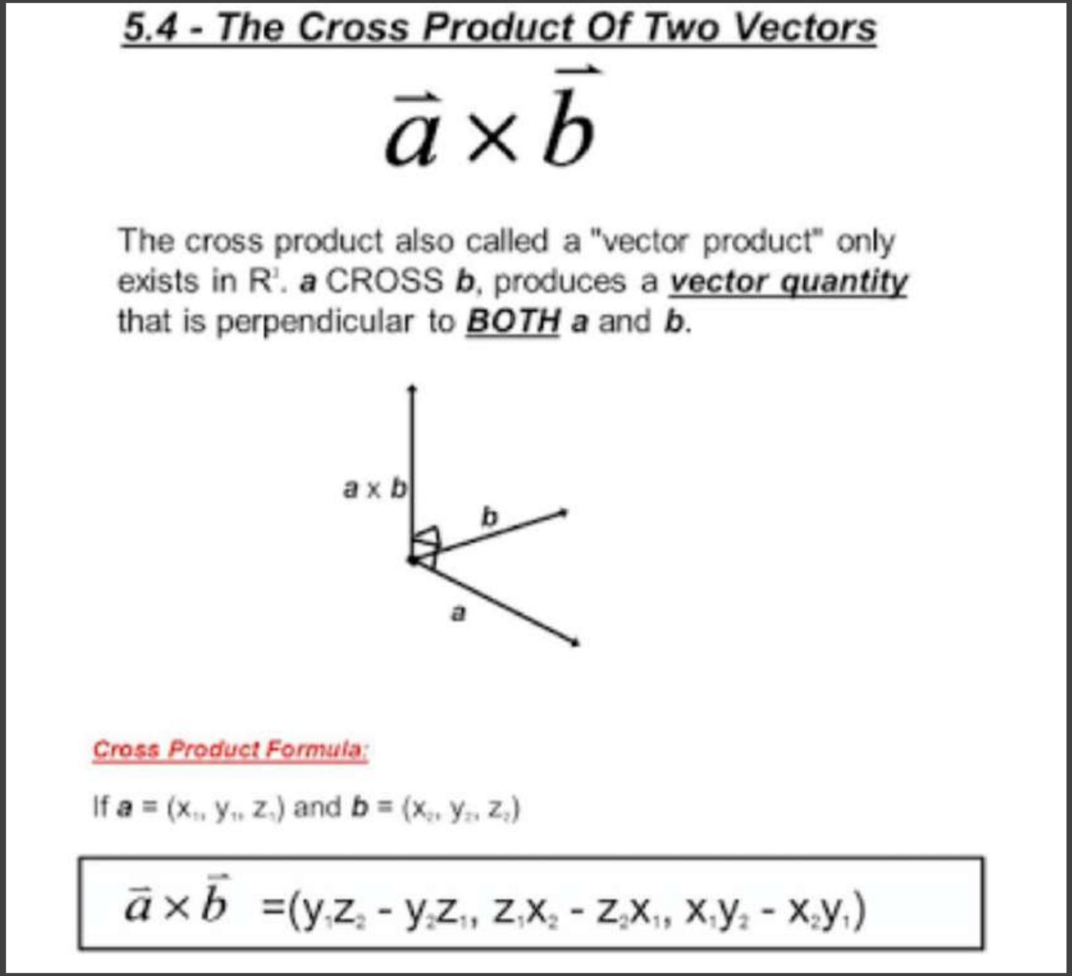
Clear your idea more about DOT PRODUCT

https://youtu.be/chdby79uD8o

https://youtu.be/0iNrGpwZwog

CROSS PRODUCT





Clear your idea more on
CROSS PRODUCT

https://youtu.be/S-lc3Uj7xH8

https://youtu.be/eu6i7WJeinw

Solved problem

8. (a) Find the unit vector perpendicular to the plane of the vectors $\vec{A}=3\hat{i}-2\hat{j}+4\hat{k}$ and $\vec{B}=\hat{i}+\hat{j}-2\hat{k}$.

(b) A force given by $\vec{F}=3\hat{i}-2\hat{j}-4\hat{k}$ is applied at the point $(1,-1,2)$. Find the moment of $\vec{F}$ about the point $(2,-1,3)$.

(c) If $\vec{A}+\vec{B}+\vec{C}=0$, prove that $\vec{A}\times\vec{B}=\vec{B}\times\vec{C}=\vec{C}\times\vec{A}$. Is the converse true?

(d) A particle moves along the curve $x=2t^2, y=t^2-4t, z=3t-5$, where t is the time. Find the components of velocity and acceleration at time $t=1$ in the direction $\hat{i}-3\hat{j}+2\hat{k}$.

[C.U. (Hons.) 1994]

(a) The resulting perpendicular vector is given by

$$\vec{C}=\pm(3\hat{i}-2\hat{j}+4\hat{k})\times(\hat{i}+\hat{j}-2\hat{k})=\pm(0\hat{i}+10\hat{j}+5\hat{k}).$$

$\therefore$ The unit vector in the direction of $\vec{C}=\dfrac{\pm(10\hat{j}+5\hat{k})}{\sqrt{10^2+5^2}}=\pm\dfrac{1}{\sqrt{5}}(2\hat{j}+\hat{k})$.

(b) The position vector of the particle at $(1,-1,2)$ with respect to $(2,-1,3)$ is given by

$$\vec{r}=(1-2)\hat{i}+(-1+1)\hat{j}+(2-3)\hat{k}=-\hat{i}-\hat{k}$$

$\therefore$ Moment of the force $\vec{F}$ about the given point is

$$\vec{M}=\vec{r}\times\vec{F}=(-\hat{i}-\hat{k})\times(3\hat{i}-2\hat{j}-4\hat{k})=-2\hat{i}-7\hat{j}+2\hat{k}$$

$\therefore$ Magnitude of the moment $=\sqrt{2^2+7^2+2^2}=\sqrt{57}$.

(c) $\because\ \vec{A}+\vec{B}+\vec{C}=0$, we may write

$$\vec{A}\times(\vec{A}+\vec{B}+\vec{C})=0 \quad\text{or}\quad \vec{A}\times\vec{B}+\vec{A}\times\vec{C}=0 \quad\therefore\ \vec{A}\times\vec{B}=\vec{C}\times\vec{A}$$

Similarly multiplying by $\vec{B}$, we can show that $\vec{B}\times\vec{C}=\vec{A}\times\vec{B}$

$$\therefore\ \vec{A}\times\vec{B}=\vec{B}\times\vec{C}=\vec{C}\times\vec{A}.$$

Converse part: If $\vec{B}\times\vec{C}=\vec{A}\times\vec{B}$, then

$$\vec{B}\times\vec{C}+\vec{B}\times\vec{A}=0 \quad\text{or}\ \vec{B}\times(\vec{A}+\vec{C})=0 \quad\text{or}\ \vec{B}\times(\vec{A}+\vec{B}+\vec{C})=0$$

$\therefore$ either $\vec{B}=0$ or $\vec{A}+\vec{B}+\vec{C}=0$.

Similarly from $\vec{B}\times\vec{C}=\vec{C}\times\vec{A}$, we can show that either $\vec{C}=0$ or $\vec{A}+\vec{B}+\vec{C}=0$ and also from $\vec{A}\times\vec{B}=\vec{C}\times\vec{A}$, we can show that either $\vec{A}=0$ or $\vec{A}+\vec{B}+\vec{C}=0$.

Thus we can conclude that either $\vec{A},\vec{B}$ and $\vec{C}$ are individually equal to zero or the sum of the vectors $(\vec{A}+\vec{B}+\vec{C})$ is a null vector.

(d) Here $\vec{r}=2t^2\hat{i}+(t^2-4t)\hat{j}+(3t-5)\hat{k}$

$$\therefore\ \frac{d\vec{r}}{dt}=4t\hat{i}+(2t-4)\hat{j}+3\hat{k} \quad\text{or}\quad \left.\frac{d\vec{r}}{dt}\right]_{t=1}=\left.\vec{v}\right]_{t=1}=4\hat{i}-2\hat{j}+3\hat{k}$$

$$\text{and}\quad \frac{d^2\vec{r}}{dt^2}=4\hat{i}+2\hat{j} \quad\text{or}\quad \left.\frac{d^2\vec{r}}{dt^2}\right]_{t=1}=\left.\vec{f}\right]_{t=1}=4\hat{i}+2\hat{j}.$$

The components of $\vec{v}$ and $\vec{f}$ at $t=1$ along the direction $(\hat{i}-3\hat{j}+2\hat{k})$ are given by the scalar products of $\vec{v}$ and $\vec{f}$ with the unit vector along $(\hat{i}-3\hat{j}+2\hat{k})$.

Thus the velocity component $=(4\hat{i}-2\hat{j}+3\hat{k})\cdot\dfrac{(\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{1^2+3^2+2^2}}=\dfrac{16}{\sqrt{14}}$ and the acceleration component $=(4\hat{i}+2\hat{j})\cdot\dfrac{(\hat{i}-3\hat{j}+2\hat{k})}{\sqrt{1^2+3^2+2^2}}=-\dfrac{2}{\sqrt{14}}$.

Assignments

4. Prove $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$.

5. If $\vec{A} = \vec{A}_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$ and $\vec{B} = B_1\hat{\imath} + B_2\hat{\jmath} + B_3\hat{k}$, prove that $\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3$.

6. Show that $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0$ and $\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$.

7. If $\vec{A}\cdot\vec{B} = 0$, show that $\vec{A}$ and $\vec{B}$ are perpendicular to each other, provided $\vec{A}$ and $\vec{B}$ are not zero.

8. Prove that $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$.

9. If $\vec{A}\times\vec{B} = 0$ and if $\vec{A}$ and $\vec{B}$ are not zero, prove that $\vec{A}$ is parallel to $\vec{B}$.

10. Show that $\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0$ and $\hat{\imath}\times\hat{\jmath} = -\hat{\jmath}\times\hat{\imath} = \hat{k}$.

11. If $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$ and $B = B_1\hat{\imath} + B_2\hat{\jmath} + B_3\hat{k}$ prove that

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

12. Show that $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$.

13. If $\vec{A} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$, $\vec{B} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$, show that $\vec{A}\times\vec{B} = -\hat{\imath} + 7\hat{\jmath} + 5\hat{k}$.