University of Calcutta
Semester -1
PHYSICS
Paper: PHS-A-CC-1-1-TH (NEW SYLLABUS)

VECTOR: SOLVED PROBLEMS And ASSIGNMENTS

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8. Find the angle between $\mathbf{A} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{B} = 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta,\quad A = \sqrt{(2)^2 + (2)^2 + (-1)^2} = 3,\quad B = \sqrt{(6)^2 + (-3)^2 + (2)^2} = 7$

$\mathbf{A}\cdot\mathbf{B} = (2)(6) + (2)(-3) + (-1)(2) = 12 - 6 - 2 = 4$

Then $\cos\theta = \dfrac{\mathbf{A}\cdot\mathbf{B}}{AB} = \dfrac{4}{(3)(7)} = \dfrac{4}{21} = 0{\cdot}1905$ and $\theta = 79^\circ$ approximately.

9. If $\mathbf{A}\cdot\mathbf{B} = 0$ and if A and B are not zero, show that $\mathbf{A}$ is perpendicular to $\mathbf{B}$.

If $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta = 0$, then $\cos\theta = 0$ or $\theta = 90^\circ$. Conversely, if $\theta = 90^\circ$, $\mathbf{A}\cdot\mathbf{B} = 0$.

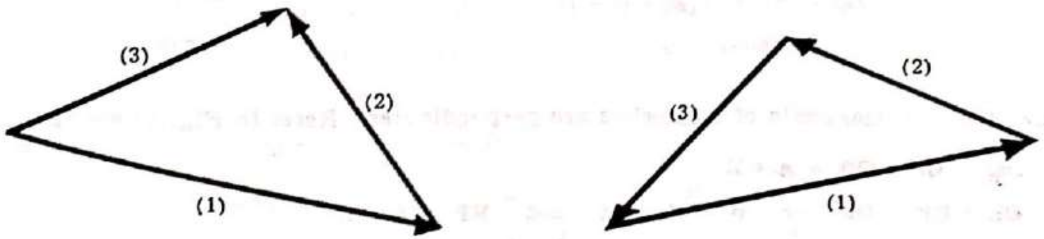
10. Determine the value of a so that $\mathbf{A} = 2\mathbf{i} + a\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ are perpendicular.

From Problem 9, $\mathbf{A}$ and $\mathbf{B}$ are perpendicular if $\mathbf{A}\cdot\mathbf{B} = 0$.

Then $\mathbf{A}\cdot\mathbf{B} = (2)(4) + (a)(-2) + (1)(-2) = 8 - 2a - 2 = 0$ for $a = 3$.

11. Show that the vectors $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{B} = \mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{C} = 2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$ form a right-angled triangle.

We first have to show that the vectors form a triangle.



(a) (b)

From the figures it is seen that the vectors will form a triangle if

(a) one of the vectors, say (3), is the resultant or sum of (1) and (2),
(b) the sum or resultant of the vectors (1) + (2) + (3) is zero,

according as (a) two vectors have a common terminal point or (b) none of the vectors have a common terminal point. By trial we find $\mathbf{A} = \mathbf{B} + \mathbf{C}$ so that the vectors do form a triangle.

Since $\mathbf{A}\cdot\mathbf{B} = (3)(1) + (-2)(-3) + (1)(5) = 14$, $\mathbf{A}\cdot\mathbf{C} = (3)(2) + (-2)(1) + (1)(-4) = 0$, and $\mathbf{B}\cdot\mathbf{C} = (1)(2) + (-3)(1) + (5)(-4) = -21$, it follows that $\mathbf{A}$ and $\mathbf{C}$ are perpendicular and the triangle is a right-angled triangle.

13. Find the projection of the vector $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ on the vector $\mathbf{B} = 4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$.

A unit vector in the direction $\mathbf{B}$ is $\mathbf{b} = \frac{\mathbf{B}}{B} = \frac{4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}}{\sqrt{(4)^2 + (-4)^2 + (7)^2}} = \frac{4}{9}\mathbf{i} - \frac{4}{9}\mathbf{j} + \frac{7}{9}\mathbf{k}$.

Projection of $\mathbf{A}$ on the vector $\mathbf{B}$ $= \mathbf{A} \cdot \mathbf{b} = (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (\frac{4}{9}\mathbf{i} - \frac{4}{9}\mathbf{j} + \frac{7}{9}\mathbf{k})$

$$= (1)(\frac{4}{9}) + (-2)(-\frac{4}{9}) + (1)(\frac{7}{9}) = \frac{19}{9}.$$

14. Prove the law of cosines for plane triangles.

From Fig.(a) below, $\mathbf{B} + \mathbf{C} = \mathbf{A}$ or $\mathbf{C} = \mathbf{A} - \mathbf{B}$.

Then $\mathbf{C} \cdot \mathbf{C} = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} - 2\mathbf{A} \cdot \mathbf{B}$

and $C^2 = A^2 + B^2 - 2AB \cos \theta$.

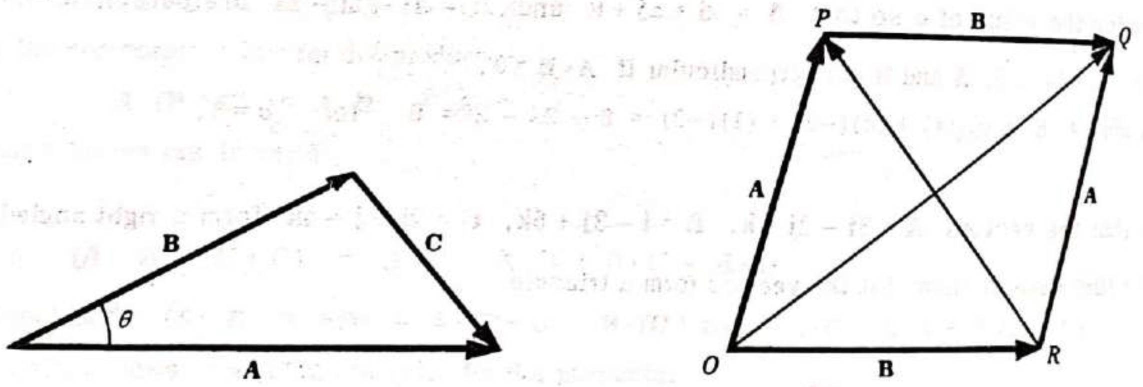


Fig.(a) Fig.(b)

15. Prove that the diagonals of a rhombus are perpendicular. Refer to Fig.(b) above.

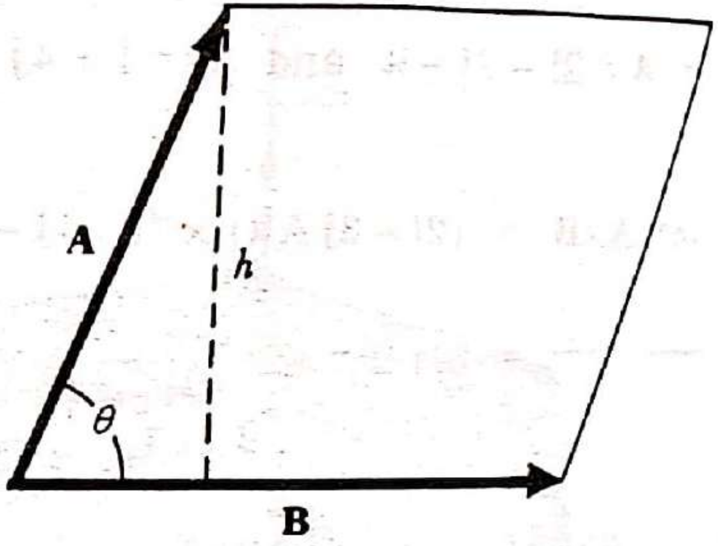
$\mathbf{OQ} = \mathbf{OP} + \mathbf{PQ} = \mathbf{A} + \mathbf{B}$

$\mathbf{OR} + \mathbf{RP} = \mathbf{OP}$ or $\mathbf{B} + \mathbf{RP} = \mathbf{A}$ and $\mathbf{RP} = \mathbf{A} - \mathbf{B}$

Then $\mathbf{OQ} \cdot \mathbf{RP} = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B}) = A^2 - B^2 = 0$, since $A = B$.

Hence $\mathbf{OQ}$ is perpendicular to $\mathbf{RP}$.

30. Prove that the area of a parallelogram with sides **A** and **B** is $|\mathbf{A} \times \mathbf{B}|$.



$$\begin{aligned} \text{Area of parallelogram} &= h\,|\mathbf{B}| \\ &= |\mathbf{A}| \sin\theta \, |\mathbf{B}| \\ &= |\mathbf{A} \times \mathbf{B}|. \end{aligned}$$

Note that the area of the triangle with sides **A** and **B** $= \frac{1}{2}|\mathbf{A} \times \mathbf{B}|$.

32. Determine a unit vector perpendicular to the plane of $\mathbf{A} = 2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ and $\mathbf{B} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

$\mathbf{A} \times \mathbf{B}$ is a vector perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = 15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}$$

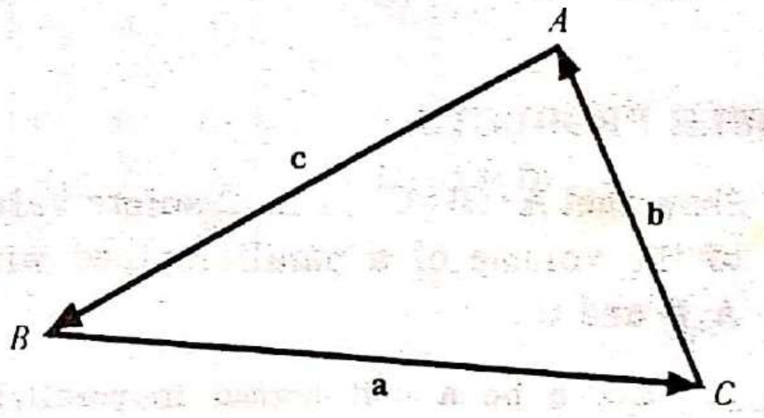
A unit vector parallel to $\mathbf{A} \times \mathbf{B}$ is $\dfrac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \dfrac{15\mathbf{i} - 10\mathbf{j} + 30\mathbf{k}}{\sqrt{(15)^2 + (-10)^2 + (30)^2}} = \dfrac{3}{7}\mathbf{i} - \dfrac{2}{7}\mathbf{j} + \dfrac{6}{7}\mathbf{k}$.

Another unit vector, opposite in direction, is $(-3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k})/7$.

Compare with Problem 16.

33. Prove the law of sines for plane triangles.

Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ represent the sides of triangle ABC as shown in the adjoining figure; then $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. Multiplying by $\mathbf{a} \times$, $\mathbf{b} \times$ and $\mathbf{c} \times$ in succession, we find



$$\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$$

i.e. $$ab \sin C = bc \sin A = ca \sin B$$

or $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

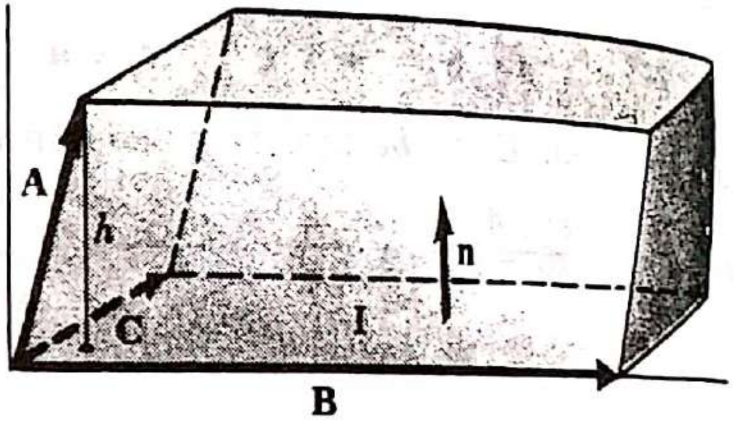
43. Prove that a necessary and sufficient condition for the vectors **A**, **B** and **C** to be coplanar is that $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = 0$.

Note that $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ can have no meaning other than $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

If **A**, **B** and **C** are coplanar the volume of the parallelepiped formed by them is zero. Then by Problem 37, $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = 0$.

Conversely, if $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = 0$ the volume of the parallelepiped formed by vectors **A**, **B** and **C** is zero, and so the vectors must lie in a plane.

37. Show that $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is in absolute value equal to the volume of a parallelepiped with sides $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$.



Let $\mathbf{n}$ be a unit normal to parallelogram I, having the direction of $\mathbf{B}\times\mathbf{C}$, and let h be the height of the terminal point of $\mathbf{A}$ above the parallelogram I.

$$\begin{aligned}\text{Volume of parallelepiped} &= (\text{height } h)(\text{area of parallelogram } I)\\ &= (\mathbf{A}\cdot\mathbf{n})(|\mathbf{B}\times\mathbf{C}|)\\ &= \mathbf{A}\cdot\{|\mathbf{B}\times\mathbf{C}|\,\mathbf{n}\} = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\end{aligned}$$

If $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ do not form a right-handed system, $\mathbf{A}\cdot\mathbf{n} < 0$ and the volume $= |\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$.

Assignment - 1

1) For what values of 'a' are $\vec{A} = a\hat{i} - 2\hat{j} + \hat{k}$ ~~and $\vec{B}$ = [illegible]~~ and $\vec{B} = 2a\hat{i} + a\hat{j} - 4\hat{k}$ perpendicular?

2) Find the projection of the vector $2\hat{i} - 3\hat{j} + 6\hat{k}$ on the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

3) If $\vec{A} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{B} = -2\hat{i} + \hat{j} - 2\hat{k}$, find a unit normal to both $\vec{A}$ and $\vec{B}$.

4) Find the work done in moving an object along a straight line from $(3, 2, -1)$ to $(2, -1, 4)$ in a force field given by $\vec{F} = 4\hat{i} - 3\hat{j} + 2\hat{k}$.