Energy of a vibrating string

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1 Energy of a vibrating string

The string posses kinetic energy due to its motion and potential energy due to its displacement against restoring forces.

The general displacement of the vibrating string at any point x at any time t is given by

$$\begin{aligned} y &= \sum_{s=1}^{\infty}(A_s \cos\frac{s\pi vt}{l} + B_s \sin\frac{s\pi vt}{l}) \sin\frac{s\pi x}{l} \\ &= \sum_{s=1}^{\infty} a_s \sin\frac{s\pi x}{l}\cos(\frac{s\pi vt}{l} - \phi_s) \end{aligned} \tag{1}$$

where $a_s = \sqrt{{A_s}^2 + {B_s}^2}$; $\phi_s = \tan^{-1}\frac{B_s}{A_s}$

1.1 Calculation of kinetic energy

If we choose an elementary length δx of the string at x whose instantaneous displacement is y , the K.E. of the element at that instant $\frac{1}{2}m\delta x(\dot{y})^2$. Hence, the kinetic energy of the whole string at the instant t is

$$W = \frac{1}{2}m\int_0^l \dot{y}^2 dx \tag{2}$$

Now

$$\dot{y} = -\frac{\pi v}{l}\sum_{s=1}^{\infty} s\ a_s \sin\frac{s\pi x}{l}\sin(\frac{s\pi vt}{l} - \phi_s)$$

$$\begin{aligned} \therefore \dot{y}^2 &= \frac{\pi^2 v^2}{l^2}\{{a_1}^2 \sin^2\frac{pix}{l}\sin^2(\frac{\pi vt}{l} - \phi_1) \\ &+ 2^2 {a_2}^2 \sin^2\frac{2\pi x}{l}\sin^2(\frac{2\pi vt}{l} - \phi_2) + ... \\ &+ s^2 {a_s}^2 \sin^2\frac{s\pi x}{l}\sin^2(\frac{s\pi vt}{l} - \phi_s) + ...\} \end{aligned}$$

+ term containing $\sin\frac{k\pi x}{l} \times \sin\frac{n\pi x}{l}$ as factors where k and n are integers but $k \neq n$. Now

$$\int_0^l \sin\frac{k\pi x}{l}\sin\frac{n\pi x}{l}dx = 0$$

and

$$\int_0^l \sin^2\frac{s\pi x}{l}dx = \frac{l}{2}$$

$$
\begin{aligned}
\therefore W &= \frac{m\pi^2 v^2}{2l^2}\{{a_1}^2\frac{l}{2}\sin^2(\frac{\pi v t}{l}-\phi_1) \\
&+ {a_2}^2 2^2\frac{l}{2}\sin^2(\frac{2\pi v t}{l}-\phi_2) + \ldots\ldots\ldots\ldots. \\
&+ {a_s}^2 s^2\frac{l}{2}\sin^2(\frac{s\pi v t}{l}-\phi_s) + \ldots\ldots\ldots\} \\
&= \frac{m\pi^2 v^2}{4l}\sum_{s=1}^{\infty} s^2{a_s}^2\sin^2(\frac{s\pi v t}{l}-\phi_s) \qquad (3)
\end{aligned}
$$

1.2 Calculation of Potential energy

Let δs be the length of the element when displaced. The workdone against the tension T to stretch the element from δx to δs is $T(\delta s-\delta x)$. This workdone is stored in the element as potential energy. Now, from figure,

$$(\delta s)^2 = (\delta y)^2 + (\delta x)^2$$

or

$$
\begin{aligned}
\delta s &= \delta x\sqrt{1+(\frac{dy}{dx})^2} \\
&= \delta x\{1+\frac{1}{2}(\frac{dy}{dx})^2\}
\end{aligned}
$$

neglecting higher order terms. Thus the Potential energy of the whole string is

$$
\begin{aligned}
V &= \frac{T}{2}\int_0^l(\frac{dy}{dx})^2dx \\
&= \frac{mv^2}{2}\int_0^l(\frac{dy}{dx})^2dx \qquad (4)
\end{aligned}
$$

But

$$\frac{dy}{dx} = \frac{\pi}{l}\sum_{s=1}^{\infty} s\ a_s\cos\frac{s\pi x}{l}\cos(\frac{s\pi v t}{l}-\phi_s)$$

$$\therefore(\frac{dy}{dx})^2 = \frac{\pi^2}{l^2}\sum_{s=1}^{\infty} s^2{a_s}^2\cos^2\frac{s\pi x}{l}\cos^2(\frac{s\pi v t}{l}-\phi_s)$$

+ terms involving $\cos\frac{s\pi x}{l}\cos\frac{\pi n x}{l}$ as factors where k and n are integers but $k = n$. Now

$$\int_0^l\cos\frac{k\pi x}{l}\cos\frac{n\pi x}{l} = 0$$

and

$$\int_0^l\cos^2\frac{s\pi x}{l}dx = \frac{l}{2}$$

Hence from equation (4)

$$
\begin{aligned}
V &= \frac{mv^2}{2}\frac{\pi^2}{l^2}\frac{l}{2}\sum_{s=1}^{\infty} s^2{a_s}^2\cos^2(\frac{s\pi v t}{l}-\phi_s) \\
&= \frac{m\pi^2 v^2}{4l}\sum_{s=1}^{\infty} s^2{a_s}^2\cos^2(\frac{s\pi v t}{l}-\phi_s) \qquad (5)
\end{aligned}
$$

$\therefore$ The total energy of the vibrating string at time t is

$$\begin{aligned} E &= T+V \\ &= \frac{m\pi^2 v^2}{4l}\sum_{s=1}^{\infty} s^2 {a_s}^2\{\sin^2(\frac{s\pi vt}{l}-\phi_s)+\cos^2(\frac{s\pi vt}{l}-\phi_s)\} \\ &= \frac{m\pi^2 v^2}{4l}\sum_{s=1}^{\infty} s^2 {a_s}^2 \qquad (6) \end{aligned}$$

Since the frequency of the s th mode of vibration is given by,

$$f_s = \frac{s}{2l}\sqrt{\frac{T}{m}} = \frac{sv}{2l}.$$

We have from (6)

$$\begin{aligned} E &= \frac{m\pi^2 v^2}{4l}\sum_{s=1}^{\infty} {a_s}^2 \frac{4l^2}{v^2} {f_s}^2 \\ &= \pi^2 ml \sum_{s=1}^{\infty} {a_s}^2 {f_s}^2 \\ &= \pi^2 M \sum_{s=1}^{\infty} {a_s}^2 {f_s}^2, \qquad (7) \end{aligned}$$

wher $M = ml$= mass of the whole string. Thus the total energy of a particular mode is proportional to the square of the frequency and the square of the amplitude of that mode of vibration.

Books Suggested:

(1). Principles of acoustics, Basudev Ghosh
(2). Sound, K. Bhattacharyya
(3). Waves and Oscillations, R. N. Chaudhuri