Damped oscillation

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1 Damped Simple Harmonic Motion(without any external force)

Let a particle of mass m undergo a SHM in a resistive medium along $x-$ direction. The restoring force acting on the particle at any instant t is proportional to the instantneous displacement of the particle, while the damping force acting on it is proportional to the instantaneous velocity of the particle. Thus if x be the displacement of the particle at time t its equation of motion is given by

$$m\frac{d^2x}{dt^2} = -k\frac{dx}{dt} - sx \tag{1}$$

, where k and s are constant. or

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0 \tag{2}$$

where, $2b = \frac{k}{m}$, $\omega^2 = \frac{s}{m}$, k is the damping factor.

Let $x = e^{\alpha t}$ be a trial solution of equation 2

$$\therefore \alpha^2 e^{\alpha t} + 2b\alpha e^{\alpha t} + \omega^2 e^{\alpha t} = 0$$

or

$$\alpha^2 + 2b\alpha + \omega^2 = 0[\because e^{\alpha t} \neq 0]$$

$$\therefore \alpha = -b \pm \sqrt{(b^2 - \omega^2)}$$

Hence the general solution of equation 2 is given by

$$\therefore x = Ae^{(-b+\sqrt{(b^2-\omega^2)}t} + Be^{(-b-\sqrt{(b^2-\omega^2)}t}$$

$$x = e^{-bt}\left[Ae^{\sqrt{(b^2-\omega^2)}t} + Be^{-\sqrt{(b^2-\omega^2)}t}\right] \tag{3}$$

where A and B are two arbitrary constants. The nature of the solution depends critically on the value of the damping coefficient b, and the behaviour is quite different depending on whether $b > \omega$, $b < \omega$ or $b = \omega$.

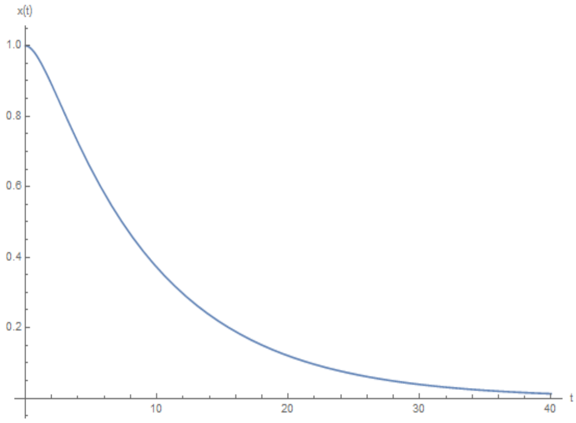

Figure 1: *An overdamped harmonic oscillator approaching equilibrium slowly.*

1.1 Special case

1.1.1 Case I: Over damped oscillations

If the damping force is large, we take $b^2 > \omega^2$. Let the particle start from a position $x = a$, where its instantaneous velocity is zero.

$\therefore$ from equation 2

$$a = A + B \tag{4}$$

Again

$$\frac{dx}{dt} = -be^{-bt}\left[Ae^{\sqrt{(b^2-\omega^2)}t} + Be^{-\sqrt{(b^2-\omega^2)}t}\right] + e^{-bt}\left[A\sqrt{b^2-\omega^2}e^{\sqrt{(b^2-\omega^2)}t} - B\sqrt{b^2-\omega^2}e^{-\sqrt{(b^2-\omega^2)}t}\right]$$

$\because \frac{dx}{dt} = 0$ at $t = 0$

$$\therefore 0 = -b\left[A + B\right] + \sqrt{b^2-\omega^2}(A - B)$$

or

$$A - B = \frac{ab}{\sqrt{b^2-\omega^2}} \tag{5}$$

from 4 and 5, we have

$$A = \frac{a}{2}\left(1 + \frac{b}{\sqrt{b^2-\omega^2}}\right) \tag{6}$$

$$B = \frac{a}{2}\left(1 - \frac{b}{\sqrt{b^2-\omega^2}}\right)$$

Thus the instantaneous displacement of the particle may be written as

$$x = e^{-bt}\left[\frac{a}{2}\left(1 + \frac{b}{\sqrt{b^2-\omega^2}}\right)e^{\sqrt{b^2-\omega^2}t} + \frac{a}{2}\left(1 - \frac{b}{\sqrt{b^2-\omega^2}}\right)e^{-\sqrt{b^2-\omega^2}t}\right] \tag{7}$$

The motion is obviously non-oscillatory or dead-beat type. The motion is started with an initial displacement but no initial velocity. The displacement gradually falls off to zero with time and the body returns to the equilibrium position without any oscillation about the equilibrium position. The graphical analysis of the vibration is depicted in Figure1.

1.1.2 Case II: Under damped oscillations

If the damping is small, we take $b^2 < \omega^2 \therefore \sqrt{b^2 - \omega^2} = i\sqrt{\omega^2 - b^2}$ where $i = \sqrt{-1}$
From the equation 3 , we get

$$
\begin{aligned}
x &= e^{-bt}\left[Ae^{i\sqrt{\omega^2-b^2}t} + Be^{-i\sqrt{\omega^2-b^2}}t\right] \\
&= e^{-bt}\left[(A+B)\cos\sqrt{\omega^2-b^2}t + i(A-B)\sin\sqrt{\omega^2-b^2}t\right]
\end{aligned}
$$

Let A_1 and A_2 be the real parts of the constants $(A+B)$ and $i(A-B)$ respectively. Then

$$
x = e^{-bt}\left[A_1\cos\sqrt{\omega^2-b^2}t + A_2\sin\sqrt{\omega^2-b^2}t\right] \tag{8}
$$

Since at $t=0$, $x=a$ and $\frac{dx}{dt}=0 \therefore$ from equation 8 $a = A_1$

$$
\therefore x = e^{-bt}\left[a\cos\sqrt{\omega^2-b^2}t + A_2\sin\sqrt{\omega^2-b^2}t\right]
$$

or

$$
\begin{aligned}
\frac{dx}{dt} &= -be^{-bt}\left[a\cos\sqrt{\omega^2-b^2}t + A_2\sin\sqrt{\omega^2-b^2}t\right] \\
&+ e^{-bt}\left[-a\sqrt{\omega^2-b^2}\sin\sqrt{\omega^2-b^2}t + A_2\sqrt{\omega^2-b^2}\cos\sqrt{\omega^2-b^2}t\right]
\end{aligned}
$$

or

$$
A_2 = \frac{ab}{\sqrt{\omega^2-b^2}}
$$

Hence equation 8 can be written as

$$
x = ae^{-bt}\left[\cos(\sqrt{\omega^2-b^2})t + \frac{b}{\sqrt{\omega^2-b^2}}\sin(\sqrt{\omega^2-b^2})t\right] \tag{9}
$$

Now let us put $1 = R\cos\theta$ and $\frac{b}{\sqrt{\omega^2-b^2}} = R\sin\theta$, where R and θ are given by $R = \frac{\omega}{\sqrt{\omega^2-b^2}}$ and $\theta = \tan^{-1}\frac{b}{\sqrt{\omega^2-b^2}}$.

$$
\therefore x = ae^{-bt}\left[R\cos\theta\cos(\sqrt{\omega^2-b^2})t + R\sin\theta\sin(\sqrt{\omega^2-b^2})t\right]
$$

or

$$
x = \frac{a\omega e^{-bt}}{\sqrt{\omega^2-b^2}}\left[\cos\left(\sqrt{\omega^2-b^2}t - \theta\right)\right] \tag{10}
$$

As evident from equation 10, the vibration of the particle is an oscillatory due to the presence of the cosine term. The amplitude of the vibration is given by $\frac{a\omega e^{-bt}}{\sqrt{\omega^2-b^2}}$. Due to the presence of the factor e^{-bt} in the amplitudes, the vibration of the article gets proressively contracted as time goes on. The time period of the oscillatory motion is given by $T = \frac{2\pi}{\sqrt{\omega^2-b^2}}$. As $t \longrightarrow \infty$, $x \longrightarrow 0$ which implies that the variation persists theoretically for an infinitely period of time, after which the amplitude reduces to zero value. The motion is graphically depicted in Figure 2.

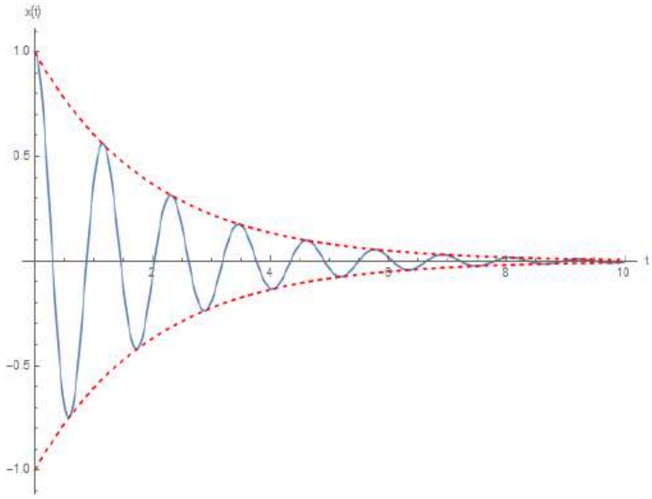

Figure 2: *Underdamped oscillations within an exponential decay envelope.*

1.1.3 Case III: Critial damping

If $b^2 \longrightarrow \omega^2$, the motion of the particle is intermediate between a dead beat type and an oscillatory type. In this transitional case $\sqrt{b^2-\omega^2}$ is very small.

Now from case I equation 7 we have

$$
\begin{aligned}
x &= \frac{a}{2}e^{-bt}\left[\left(1+\frac{b}{\sqrt{b^2-\omega^2}}\right)e^{(\sqrt{b^2-\omega^2})t}+\left(1-\frac{b}{\sqrt{b^2-\omega^2}}\right)e^{-(\sqrt{b^2-\omega^2})t}\right] \\
&= \frac{a}{2}e^{-bt}\left[\left(e^{(\sqrt{b^2-\omega^2})t}+e^{-(\sqrt{b^2-\omega^2})t}\right)+\frac{b}{\sqrt{b^2-\omega^2}}\left(e^{(\sqrt{b^2-\omega^2})t}+e^{-(\sqrt{b^2-\omega^2})t}\right)\right] \\
&= \frac{a}{2}e^{-bt}\left[\left(1+\sqrt{b^2-\omega^2}t+1-\sqrt{b^2-\omega^2}t\right)+\frac{b}{\sqrt{b^2-\omega^2}}\left(1+\sqrt{b^2-\omega^2}t-1+\sqrt{b^2-\omega^2}t\right)\right]
\end{aligned}
$$

Neglecting the higher order terms

$$
\begin{aligned}
x &= e^{-bt}(a+abt) \\
&= e^{-bt}(A+Bt) \qquad (11)
\end{aligned}
$$

where we put $a = A; ab = B$

Equation 11 represents a critically damped vibration which is intermediate between the cases studied above. Figure 3 shows this situation is also non-oscillatory. But here the decay is much faster than the overdamped case. The motion is now said to be critically damped.

There are two physical effects play in a damped oscillator. The first is the damping which tries to bring any motion to stop. This operates on a time scale $T_d \approx 1/b$. The restoring force exerted tries to make the system oscillate and this operates on a time-scale $T_0 = 1/\omega$. We have overdamped oscillations if the damping operates on a shorter time-scale compared to the oscillations $i.e. T_d < T_0$ which completely destroyes the oscillatory behaviour.

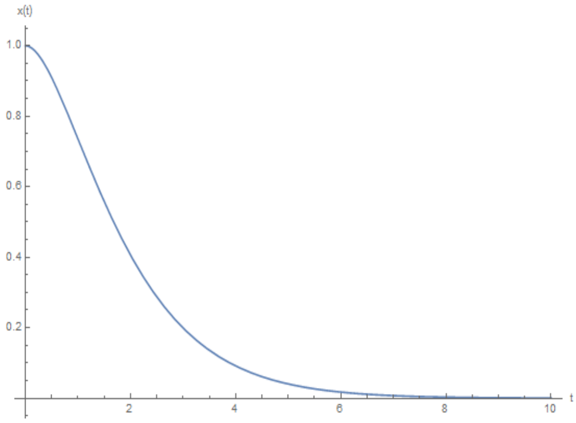

Figure 3: *Time evolution of the amplitude of a critically damped harmonic oscillator.*

2 Damped LC oscillations (LCR circuit)

If resistance R is present in an LC circuit, the total energy

$$U = \frac{1}{2}Li^2 + \frac{q^2}{2C}$$

is no longer constant, but decreases with time as it is transformed steadily to thermal energy in the resistor:

$$\frac{dU}{dt} = -i^2R$$

Hence

$$Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = -i^2R$$

Substituting $i = \frac{dq}{dt}$ and $\frac{di}{dt} = \frac{d^2q}{dt^2}$ and dividing by i, we get

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

$$\ddot{q} + 2b\dot{q} + \omega^2 q = 0 \tag{12}$$

which is the differential equation that describes the damped oscillations.

The general solution of equation 12 can be written as

$$q = Qe^{-bt}\cos(\omega' t - \theta). \tag{13}$$

Here $b = R_{\overline{2L}}$, $\omega = \frac{1}{\sqrt{Lc}}$, $\omega' = \sqrt{\omega^2 - b^2}$.

3 Assignment II

1. Obtain solution for critical damping as a limiting case ($b \longrightarrow \omega$ of overdamped solution.
2. An under-damped oscillator has a time period of 2s and the amplitude of oscillation goes down by 10% in one oscillation.
(a) what is the logarithmic decrement of the oscillator?
(b) Determine the damping coefficient b.
(c) What would be the time period of this oscillator if there was no damping?
(d) What should be b if the time period is to be increased to 4s?