Forced Oscillations

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1 Mathematical analysis of Forced vibration

Let us consider a system of mass m which is allowed to execute a periodic motion along $x-$ axis under the influence of an external periodic force $Fsinpt$ of amplitude F and angular frequency p. The natral angular frequency of the driven system is ω. Further the system is subjected to a restoring force proportional to the instantaneous displacement and a damping force is proportional to the instantaneous velocity of the system. Hence the equation of the motion of the system at any time t is given by

$$m\frac{d^2x}{dt^2} = -sx - k\frac{dx}{dt} + F\sin pt$$

, where s and k are constant. or

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = f\sin pt \tag{1}$$

where, $2b = \frac{k}{m}$, $\omega^2 = \frac{s}{m}$; $f = \frac{p}{m}$

Let x_1 be the C.F. of equation 1,

$$\frac{d^2x_1}{dt^2} + 2b\frac{dx_1}{dt} + \omega^2 x_1 = 0 \tag{2}$$

Let $x_1 = e^{\alpha t}$ be a trial solution of equation 2

$$\therefore \alpha^2 e^{\alpha t} + 2b\alpha e^{\alpha t} + \omega^2 e^{\alpha t} = 0$$

or

$$\alpha^2 + 2b\alpha + \omega^2 = 0[\because e^{\alpha t} \neq 0]$$

$$\therefore \alpha = -b \pm \sqrt{(b^2 - \omega^2)}$$

Hence the general solution of equation 2 is given by

$$\therefore x = Pe^{(-b+\sqrt{(b^2-\omega^2)}t} + Qe^{(-b-\sqrt{(b^2-\omega^2)}t}$$

$$x = e^{-bt}\left[Pe^{\sqrt{(b^2-\omega^2)}t} + Qe^{-\sqrt{(b^2-\omega^2)}t}\right] \tag{3}$$

where P and Q are two arbitrary constants.

Since the system is always under the influence of a driving force, the damping suffered by the system may be assumed to be small. Thus we take $b^2 < \omega^2$.

$\therefore$ from equation 3

$$x = e^{-bt}\left[Pe^{i\sqrt{(\omega^2-b^2)}t} + Qe^{-i\sqrt{(\omega^2-b^2)}t}\right]$$

or

$$x_1 = e^{-bt}\left[(P+Q)\cos\sqrt{\omega^2-b^2}t + i(P-Q)\sin\sqrt{\omega^2-b^2}t\right]$$

If A_1 and A_2 be the real parts of $(P+Q)$ and $i(P-Q)$ respectively, we have

$$x_1 = e^{-bt}\left[A_1\cos\sqrt{\omega^2-b^2}t + A_2\sin\sqrt{\omega^2-b^2}t\right]$$

Let us put $A_1 = a\sin\theta$ and $A_2 = a\cos\theta$ where $a = \sqrt{{A_1}^2 + {A_2}^2}$ and $\theta = \tan^{-1}\frac{A_1}{A_2}$

$$x_1 = e^{-bt}\left[a\sin(\sqrt{\omega^2-b^2}t+\theta)\right] \tag{4}$$

This x_1 repesents the natural vibration of the system, which vanishes after a short interval of time.

The P.I. of equation 1 can be witten as

$$x_2 = A\sin(pt-\alpha) \tag{5}$$

where A and α are two constants to be determined later.

We can make this supposition of the ground that the system will ultimately vibrate with the same frequency (p) as that of the driver.

from equation 1

$$-Ap^2\sin(pt-\alpha) + 2bAp\cos(pt-\alpha) + \omega^2A\sin(pt-\alpha) = f\sin(pt-\alpha+\alpha)$$

or

$$A(\omega^2-p^2)\sin(pt-\alpha) + 2bAp\cos(pt-\alpha) = f\sin(pt-\alpha)\cos\alpha + f\cos(pt-\alpha)\sin\alpha \tag{6}$$

Comparing the co-efficients of $\sin(pt-\alpha)$ and $\cos(pt-\alpha)$ from both sides of the equation6 we get

$$f\cos\alpha = A(\omega^2-p^2)$$

and

$$f\sin\alpha = 2bAp$$

$$\therefore f^2 = A^2\left[\left(\omega^2-p^2\right)^2 + 4b^2p^2\right]$$

or

$$A = \frac{f}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}} \tag{7}$$

and

$$\alpha = \tan^{-1}\left(\frac{2bp}{\omega^2 - p^2}\right) \tag{8}$$

The complete solution of equation 1 is

$$\begin{aligned} x &= x_1 + x_2 \\ &= ae^{-bt}\sin(\sqrt{\omega^2 - b^2}t + \theta) + \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4b^2p^2}}\sin(pt - \alpha) \end{aligned} \tag{9}$$

where $\theta = \tan^{-1}\left(\frac{A_1}{A_2}\right)$ and $\alpha = \tan^{-1}\left(\frac{2bp}{\omega^2 - p^2}\right)$.

The first part of the solution 9 will persists for a short interval after which it will die out. Then the second term becomes pedominant and hence in the steady state the system will vibrate with the same frequency as that of the driver.

Since, the natural vibration of the system persists for a small time interval at the initial stage, the linear frequency $\frac{p}{2\pi}$ of the driver and the natural frequency $\frac{\sqrt{\omega^2 - b^2}}{2\pi}$ of the driven sysem will interfare with each other to produce beats, provided these two frequencies are nearly equal. Since theses beats survive for a very short time interval at the initial stage, they are kown as 'transient beats'.

When a forced system comes to a steady state of motion, its frequency of the vibrations becomes equal to that of the driver. Thus in the steady state the natural angular frequency(ω) of the driven system dies out and the steady state motion of the system is given by, $x = A\sin(pt - \alpha)$, where p is the angular frequency of the driver and α is the phase difference between the driver and the driven systems.

Books Suggested:

(1). Principles of acoustics, Basudev Ghosh
(2). Sound, K. Bhattacharyya
(3). Waves and Oscillations, R. N. Chaudhuri