Response and Resonance

Dr. Soma Mandal,
Assistant professor,
Department of Physics, Government Girls' General Degree College, Kolkata

1 Resonance and sharpness of resonance

In the steady state, the instantaneous displacement of the system is given by

$$x = x_2 = A\sin(pt - \alpha)$$

Kinetic energy of the system at time t is

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2p^2\cos^2(pt - \alpha)$$

Since, in a particular motion the total energy of a system remains constant which is equal to the maximum of its kinetic energy. We may write

total energy $= E = (K.E)_{max} = \frac{1}{2}A^2p^2$

or

$$\begin{aligned} E &= \frac{1}{2}mp^2\frac{f^2}{(\omega^2 - p^2)^2 + 4b^2p^2} \\ &= \frac{\frac{1}{2}mf^2}{\omega^2\left(\frac{\omega}{p} - \frac{p}{\omega}\right)^2 + 4b^2} \end{aligned}$$

If $p = \omega$, the energy of the system is maximum for any given value of b. Thus when frequency of the driver coincides with the natural frequency of the driven, the energy of the driven system is maximum. This phenomenon is known as velocity resonance or energy resonance or resonance. Writing $\Delta = \frac{\omega}{p} - \frac{p}{\omega}$=mistuning between the driver and the driven.

$$E = \frac{\frac{1}{2}mf^2}{\omega^2\Delta^2 + 4b^2} \tag{1}$$

E is called the energy of response of the driven system. Evidently, E will be maximum when $\Delta = 0$.

$$E = \frac{\frac{1}{2}mf^2}{4b^2} \tag{2}$$

This E_m is called the energy of resonance.

As evident from equation 1, the energy of response correspong to a particular mistuning (Δ) is larger for a smaller value of the damping of the medium. The graphical analysis of E versus Δ is shown in Figure 1.

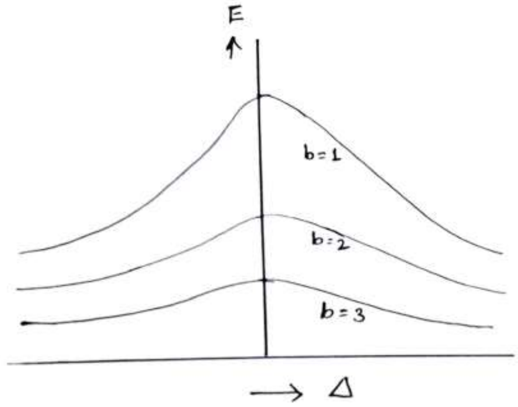

Figure 1: *Response curves for different values of b*

1.1 Sharpness of resonance

When $\Delta = 0$, $E = E_m$, and the situation is called velocity (or energy) resonance or simply resonance. In order to study how sharp the resonances, we must consider the factors controlling resonance. Quantitatively , the sharpness of resonance (S) is defined as the reciprocal of the mistuning (Δ) at which the energy of response is half that at resonance. Now

$$\frac{E_m}{E} = \frac{\omega^2\Delta^2 + 4b^2}{4b^2} = 1 + \frac{\omega^2\Delta^2}{4b^2}$$

$\because \frac{E}{E_m} = \frac{1}{2}$

$$\therefore \frac{E_m}{E} = 2 = 1 + \frac{\omega^2\Delta^2}{4b^2}$$

or

$$\frac{\omega^2\Delta^2}{4b^2} = 1$$

or

$$\frac{1}{\Delta} = \pm\frac{\omega}{2b}$$

$$S = \pm\frac{\omega}{2b} \tag{3}$$

It is evident from equation 3 that the sharpness of resonance, is larger for smaller values of b.

The following curves (shown in Figure 2)represent the sharpness of resonance for different values of b.

As evident from the above curves, $\frac{E}{E_m}$ or $\frac{E_m}{E} \longrightarrow 1$ if $b \longrightarrow \infty$ for any mistuning (Δ). Hence the resonance in this case is entirely flat.As when b is small the curves are very sharp near resonance. As b increases, the sharpness decreases and the resonance tries to become flat.

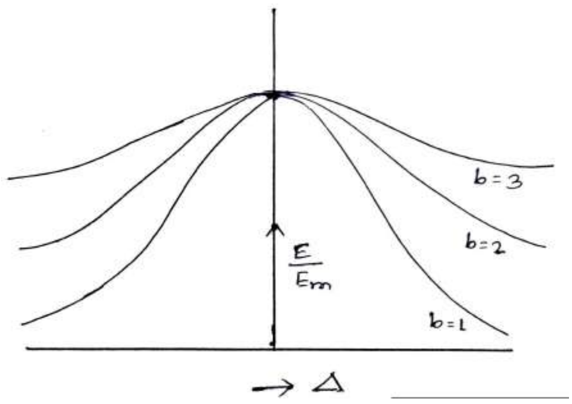

Figure 2: *Sharpness of resonance for different values of b*

1.2 Quality factor Q

Quantitatively the sharpness of resonance is measured in terms of quality factor Q which is defined by $Q = \frac{Resonant frequency}{Bandwidth}$.

we have

$$\Delta = \frac{\omega}{p} - \frac{p}{\omega}$$

Suppose the angular frequency of the force is slightly larger than ω; then at that angular frequency $p = \omega + \delta p$, Δ is given by

$$\begin{aligned} \Delta &= \frac{\omega}{\omega + \delta p} - \frac{\omega + \delta p}{\omega} \\ &= \left(1 + \frac{\delta p}{\omega}\right)^{-1} - \left(1 + \frac{\delta p}{\omega}\right) \\ &= -2\frac{\delta p}{\omega} \end{aligned}$$

as δp is very small. Suppose a amall change in p from ω to $\omega + \delta p$ or $\omega - \delta p$ causes $\frac{E}{E_m}$ fall to half. Then

$$\frac{1}{\Delta} = \pm\omega 2b = \frac{\omega}{2\delta p} = \pm Q$$

where Q is the ratio of frequency at resonance to the difference in frequencies at points where power dissipation decreases to half that at resonance. Q is called the quality factor.

$$Q = \frac{1}{\Delta} = \frac{\omega}{2b} = \frac{mn}{k}.$$

1.3 Problem 1

In the steady state forced vibration describe how the phase of the driven system changes with frequency of the driving system.

Solution: The phase difference between the driver and the driven system is given by

$$\alpha = \tan^{-1}\left(\frac{2bp}{\omega^2 - p^2}\right)$$

so that

$$\sin\alpha = \frac{2bAP}{f}$$

and

$$\cos\alpha = \frac{A(\omega^2 - p^2)}{f}$$

(i) If $p < \omega$, both $\sin\alpha$ and $\tan\alpha$ are positive $\Rightarrow 0 < \alpha < \frac{\pi}{2}$.

(ii) If $p > \omega$, $\sin\alpha$ is positive but $\tan\alpha$ is negative. $\Rightarrow \frac{\pi}{2} < \alpha < \pi$.

(iii) If $p \to \infty$, $\tan\alpha \to 0$, $\sin\alpha \to 0$. Hence $\alpha \to \pi$. Thus for any value of p, α lies between 0 and π.

(iv) If $p = \omega$, $\alpha = \frac{\pi}{2}$ (at resonance). Thus at velocity resonance the driven system lags behind the driver by an angle $\frac{\pi}{2}$.

(v) When $p = 0$, $\alpha = 0$. There is no difference of phase between the driven system and the impressed force.

$$\begin{aligned}
\therefore \frac{d\alpha}{dp} &= \frac{1}{1 + \left(\frac{2bp}{\omega^2 - p^2}\right)^2} \frac{(\omega^2 - p^2).2b + 2bp.2p}{(\omega^2 - p^2)^2} \\
&= \frac{1}{1 + \frac{4b^2p^2}{(\omega^2 - p^2)^2}} \frac{(\omega^2 + p^2).2b}{(\omega^2 - p^2)^2} \\
&= \frac{(\omega^2 + p^2)2b}{(\omega^2 - p^2)^2 + 4b^2p^2}
\end{aligned}$$

$$\therefore \frac{d\alpha}{dp}_{at\ resonance} = \frac{2\omega^2.2b}{4b^2\omega^2} = \frac{1}{b} \qquad [\because p = w]$$

It is evident that the rate of change of α with p is larger at resonance for a medium with small damping co-efficient and vice-verse. This is illustrated in Figure 3.

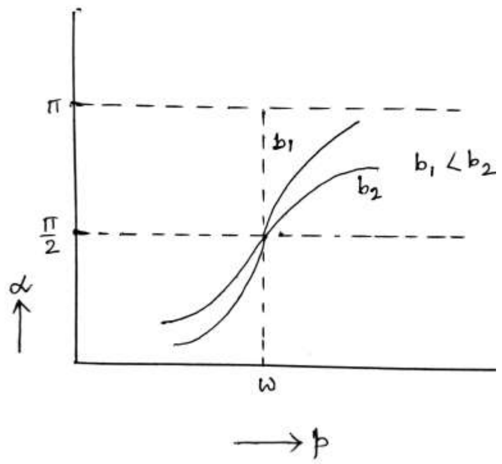

Figure 3: *The variation of α with p*

1.4 Power dissipation

Show that the rate of supply of energy by the driver is equal to the the rate of dissipation of energy by the driven system against the damping force in the steady state.

Let us assume that the driving force $Fsinpt$ is displaced through an elementay length dx in time dt after t. Hence the work done by the force in time dt is $Fsinpt.dx$. If T be the periodic time of vibration, the average rate of supply of energy by the driver is

$$W_1 = \frac{1}{T}\int_0^T F\sin pt\frac{dx}{dt}dt$$

Now

$$x = A\sin(pt-\alpha)$$

$$\therefore \frac{dx}{dt} = Ap\cos(pt-\alpha)$$

$$\begin{aligned}
\therefore W_1 &= \frac{1}{T}FAp\int_0^T F\sin pt\cos(pt-\alpha)dt \\
&= \frac{FAp}{T}\left[\int_0^T \sin pt\cos(pt)\cos\alpha dt + \int_0^T \sin^2 pt\sin\alpha dt\right] \\
&= \frac{FAp}{T}\left[0+\frac{T}{2}\sin\alpha\right] \\
&= \frac{FAp}{2}\sin\alpha
\end{aligned}$$

Now,

$$\sin\alpha = \frac{2bAp}{f} \tag{4}$$

$$\begin{aligned}\therefore W_1 &= \frac{FAp}{2}\frac{2bAp}{f} \\ &= \frac{FA^2p^2b}{f} \\ &= mA^2p^2\frac{k}{2m} \qquad \left[\because \frac{F}{f} = m; 2b = \frac{k}{m}\right] \\ &= \frac{1}{2}kA^2p^2 \end{aligned} \tag{5}$$

Again, let us assume that the driven system undergoes an elementary displacement δx in a time interval δt after t against the frictional force of the medium. Hence the workdone against the damping in time δt is $k\frac{dx}{dt}.\delta x$.

Therefore average rate of dissipation of energy against the frictional force in a complete period T is

$$\begin{aligned} W_2 &= \frac{1}{T}\int_0^T k\frac{dx}{dt}\frac{dx}{dt}dt \\ &= \frac{k}{T}A^2p^2\int_0^T \cos^2(pt-\alpha)dt \\ &= \frac{k}{T}A^2p^2\frac{T}{2} \\ &= \frac{1}{2}kA^2p^2 \end{aligned} \tag{6}$$

From equation 5 and 6 $W_1 = W_2$. Hence the theorem is established.

Books Suggested:

(1). Principles of acoustics, Basudev Ghosh
(2). Sound, K. Bhattacharyya
(3). Waves and Oscillations, R. N. Chaudhuri