Superposition of two collinear Harmonic Oscillations

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1 Principle of superposition

The principle of superposition states that " The resultant of two or more harmonic displacements is simply the vector sum of the individual displacements".

For a linear homogeneous differential equationm, the sum of any two solutions is itself a solution.

Let us consider a linear homogeneous differential equation of degree n:

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + ... + a_1\frac{dy}{dt} + a_0 y = 0$$

If y_1 and y_2 are two solutions of this equation then $y_1 + y_2$ is also a solution.

For a driven harmonic oscillator

$$m\frac{d^2x}{dt^2} = -kx + F(t)$$

where $F(t)$ is the external force which is independent of x. Suppose that a driving force $F_1(t)$ produces an oscillation $x_1(t)$ and another driving force $F_2(t)$ produces an oscillation $x_2(t)$. When the total driving force is $F_1(t) + F_2(t)$, the corresponding oscillation is given by $x(t) = x_1(t) + x_2(t)$.

1.1 Superposition of two collinear SHMs of same frequency but having different amplitude and phases

Let two SHMs be represented by

$$x_1 = a_1 \cos(\omega t + \delta_1) \tag{1}$$
$$x_2 = a_2 \cos(\omega t + \delta_2) \tag{2}$$

where $a_1$and a_2 are the amplitudes, δ_1 and δ_2 are the initial phase angles of the two SHMs of same angular frequency ω.

1.1.1 Analytical method

By the superposition principle the resultant displacement is given by

$$\begin{aligned} x &= x_1 + x_2 \\ &= a_1\cos(\omega t + \delta_1) + a_2\cos(\omega t + \delta_2) \\ &= (a_1 cos\delta_1 + a_2\cos\delta_2)\cos\omega t - (a_1\sin\delta_1 + a_2\sin\delta_2)\sin\omega t \end{aligned} \tag{3}$$

Putting

$$a_1 cos\delta_1 + a_2\cos\delta_2 = A\cos\phi \tag{4}$$

$$a_1 \sin \delta_1 + a_2 \sin \delta_2 = A \sin \phi \tag{5}$$

we get

$$x = A \cos(\omega t + \phi) \tag{6}$$

It shows that the motion is always simple harmonic. The constants A and ϕ can be found out from equation 4 and 5. Squaring and adding these equations we get

$$A^2 = {a_1}^2 + {a_2}^2 + 2a_1 a_2 \cos(\delta_1 - \delta_2) \tag{7}$$

Dividing equation 5 by 4, we get

$$\tan \phi = \frac{a_1 \sin \delta_1 + a_2 \sin \delta_2}{a_1 \cos \delta_1 + a_2 \cos \delta_2} \tag{8}$$

If instead of two SHMs there are several SHMs of same frequency but different amplitudes and phases then equations 4 and 5 take the form

$$\sum_i a_i \cos \delta_i = A \cos \phi$$

$$\sum_i a_i \sin \delta_i = A \sin \phi$$

In this case also the resultant motion is a SHM given by

$$x = Acos(\omega t + \phi)$$

where

$$A^2 = \left(\sum_i a_i \sin \delta_i \right)^2 + \left(\sum_i a_i \cos \delta_i \right)^2$$

and

$$\tan \phi = \frac{\sum_i a_i \sin \delta_i}{\sum_i a_i \cos \delta_i}$$

1.1.2 Vector method

The rotating vector representation of SHM provides a simple method of obtaining the resultant of SHMs of same frequency. Let us represent the first SHM by a rotating vector $\overrightarrow{OB_1}$ of constant length a_1 rotating anticlockwise with a constant angular velocity ω and making an angle $\omega t + \delta_1$ with the $x-$ axis at any time t. The projection ON_1 of this vector on $x-$ axis gives the displacement x_1 at any time t. Similarly the rotating vector $\overrightarrow{OB_2}$ will represent the second SHM. Now the resultant motion will be given by the vector sum of $\overrightarrow{OB_1}$ and $\overrightarrow{OB_2}$. By parallelogram of vector addition the magnitude A of the resultant $\overrightarrow{OB}$ is given by

$$A^2 = {a_1}^2 + {a_2}^2 + 2a_1 a_2 \cos(\delta_1 - \delta_2)$$

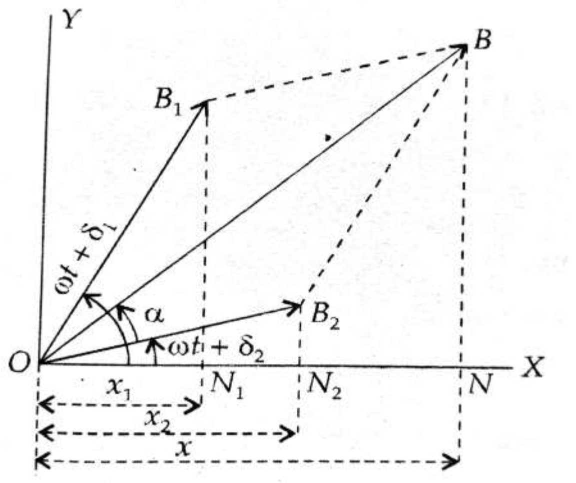

Figure 1: *Two vectors representing SHMs along with the resultant vector*

If the resultant $\overrightarrow{OB}$ makes an angle $\omega t + \delta$ with $x-$ axis then $\delta = \delta_2 + \alpha$.

$$\therefore \tan\delta = \frac{\tan\delta_2 + \tan\alpha}{1 - \tan\delta_2 \tan\alpha}$$

Now

$$\tan\alpha = \frac{a_1 \sin(\delta_1 - \delta_2)}{a_2 + a_1 \cos(\delta_1 - \delta_2)}$$

Substituting $\tan\alpha$ in the above equation and simplifying we get

$$\tan\delta = \frac{a_1 \sin\delta_1 + a_2 \sin\delta_2}{a_1 \cos\delta_1 + a_2 \cos\delta_2}$$

The projection of $\overrightarrow{OB}$ on $x-$ axis is

$$x = A\cos(\omega t + \delta)$$

which represents a simple harmonic motion.

2 Two SHMs of slightly different frequencies acting along the same direction: Beats

Let us consider two SHMs having slightly different angular frequencies ω and $\omega + \Delta\omega$ where $\Delta\omega << \omega$.

$$\begin{aligned} x_1 &= a_1 \cos(\omega t + \delta_1) \\ x_2 &= a_2 \cos[(\omega + \Delta\omega)t + \delta_2] \\ &= a_2 \cos(\omega t + {\delta_2}') \end{aligned}$$

where ${\delta_2}' = \Delta\omega t + \delta_2$.

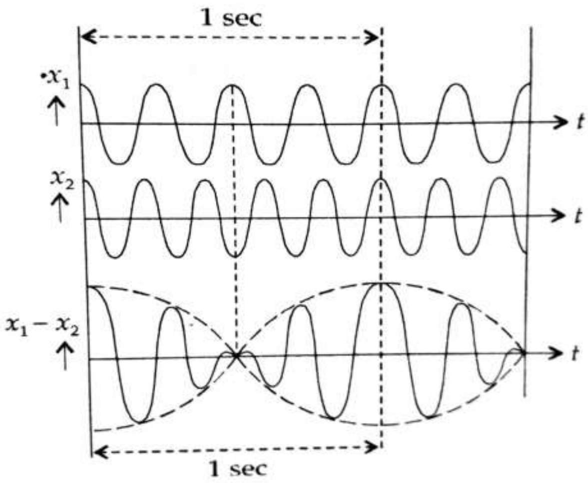

Figure 2: *Superposition of SHMs with slightly different frequencies*

The resultant displacement is given by

$$\begin{aligned} x &= x_1 + x_2 \\ &= (a_1 \cos\delta_1 + a_2 \cos{\delta_2}') \cos\omega t - (a_1 \sin\delta_1 + a_2 \sin{\delta_2}') \sin\omega t \end{aligned}$$

Putting

$$a_1 \cos\delta_1 + a_2 \cos{\delta_2}' = A\cos\phi$$
$$a_1 \sin\delta_1 + a_2 \sin{\delta_2}' = A\sin\phi$$

we get

$$x = Acos(\omega t + \phi)$$

where

$$\begin{aligned} A^2 &= {a_1}^2 + {a_2}^2 + 2a_1a_2\cos(\delta_1 - {\delta_2}') \\ &= {a_1}^2 + {a_2}^2 + 2a_1a_2\cos(\Delta\omega t + \delta_2 - \delta_1) \end{aligned}$$

and

$$\tan\phi = \frac{a_1 \sin\delta_1 + a_2 \sin{\delta_2}'}{a_1 cos\delta_1 + a_2 \cos{\delta_2}'}$$

The resultant motion is not simple harmonic because both the amplitude A and phase constant ϕ very with time. As the time increases the amplitude A attains maximum value $a_1 + a_2$ when $\cos(\Delta\omega t + \delta_2 - \delta_1) = 1$ or $\Delta\omega t + \delta_2 - \delta_1 = 0, 2\pi, 4\pi, ...$ etc. The amplitudes attains minimum value $a_1 \sim a_2$ when $\cos(\Delta\omega t + \delta_2 - \delta_1) = -1$ or $\Delta\omega t + \delta_2 - \delta_1 = \pi, 3\pi, 5\pi, ...$ etc. Thus the amplitude of the resultant vibration changes periodically with a frequency equal to $\frac{\Delta\omega}{2\pi}$ which is equal to the difference in frequencies of the component vibrations.

This phenomenon is known as beats. and is observed when two tuning forks or any two sources of sound of nearly equal frequencies are sounded together. The number of beats per second equals the difference of frequencies of the component waves.

Figure 2 diplays graphically the result of superposition of two SHMs of slightly different frequencies. In this figure we show superposition of two SHMs having frequencies of $4Hz$ and $5Hz$. They are further assumed to have same amplitudes and same initial phases. The time interval between two successive maxima or minima in the resultant pattern in this case is 1 sec. The resultant amplitude thus varies with a frequency of 1 Hz which is equal to the difference of frequencies of the component vibrations.

3 Assignment

1. Two simple harmonic motions of same angular frequency ω

$$\begin{aligned} x_1 &= a_1 \sin \omega t \\ x_2 &= a_2 \sin(\omega t + \phi) \end{aligned}$$

act on a particle along the $x-$ axis simultaneously. Find the resultant motion.

2. Two vibrations along the same line are described by the equations

$$\begin{aligned} x_1 &= 0.03 \cos(10\pi t) \\ x_2 &= 0.03 \cos(12\pi t) \end{aligned}$$

where x_1 , x_2 are measured in meters and t in seconds. Obtain the equation describing the resultant motion and hence find the beat frequency.

3. Two adjacent piano keys are struck simultaneously. The notes emitted by them have frequencies ν_1 and ν_2. Find the number of beats heard per second.

Books Suggested:
(1). Principles of acoustics, Basudev Ghosh
(2). Sound, K. Bhattacharyya
(3). Waves and Oscillations, R. N. Chaudhuri