Boundary conditions and amplitude of Plucked String

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1 Plucked string

We consider a string of length l rigidly clamped at its end points $x = 0$ and $x = l$. Hence the string is stretched along $x-$ axis. The string is excited to transverse vibration of plucking it at some point C (where AC=h) through a vertical height CD=k. As soon as it is released at D, tranverse progressive waves will travel from the point of excitation along two opposite directions. These waves will be reflected from the end point approaches each other and superpose to produce stationary waves. The initial configuration of the string is shown in the diagram.

The general displacement of any point x at any time t is given by

$$y = \sum_{s=1}^{\infty} (A_s \cos \frac{s\pi vt}{l} + B_s \sin \frac{s\pi vt}{l}) \sin \frac{s\pi x}{l} \tag{1}$$

where A_s's and B_s's are Fourier constants and v is the velocity of propagation of transverse wave along the length of the string. The boundary conditions require that $y = 0$ at $x = 0$ and l for all values of t. The instantaneous velocity at the point x is

$$\dot{y} = \sum_{s=1}^{\infty} \left(-A_s \frac{s\pi v}{l} \sin \frac{s\pi vt}{l} + B_s \frac{s\pi v}{l} \cos \frac{s\pi vt}{l} \right) \sin \frac{s\pi x}{l}$$

Since at time $t = 0$ each point on the string is at rest, we have

$$(\dot{y})_0 = 0 = \sum_{s=1}^{\infty} B_s \frac{s\pi v}{l} \sin \frac{s\pi x}{l}$$

$\Rightarrow B_s$'s are zero.

The equation (1) can be reduced to

$$y = \sum_{s=1}^{\infty} A_s \cos \frac{s\pi vt}{l} \sin \frac{s\pi x}{l} \tag{2}$$

$$y_0 = \sum_{s=1}^{\infty} A_s \sin \frac{s\pi x}{l} \tag{3}$$

In order to evaluate A_s, we multiply both sides of equation 3 by $\sin \frac{m\pi x}{l}$ (where m is an integer) and integrate the result with respect to x from $x = 0$ to $x = l$.

$$\begin{aligned} \therefore \int_0^l y_0 \sin \frac{m\pi x}{l} dx &= \int_0^l \sum_{s=1}^{\infty} (A_s \sin \frac{s\pi x}{l}) \sin \frac{m\pi x}{l} dx \\ &= 0, \; if \; s \neq m \\ &= A_m . \frac{l}{2} \; if \; s = m \end{aligned}$$

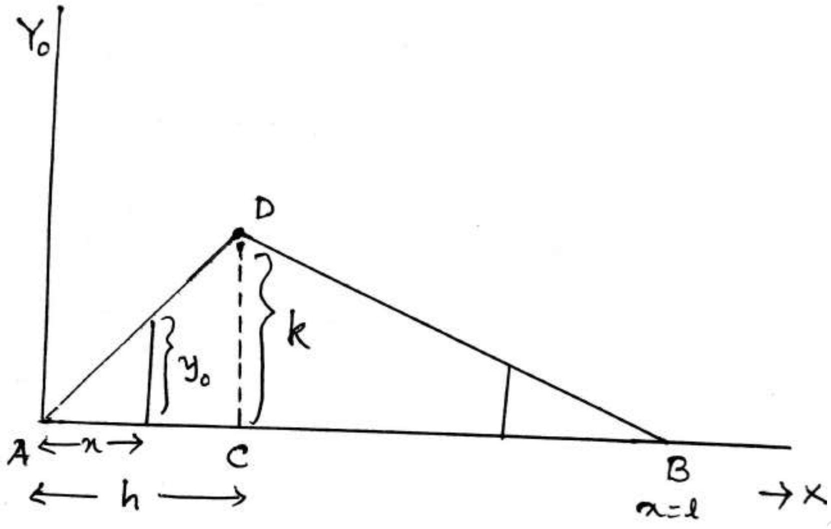

Figure 1: *Plucked string*

$$\therefore \int_0^l y_0 \sin\frac{m\pi x}{l}dx = A_m\frac{l}{2}$$

$$\therefore A_m = \frac{2}{l}\int_0^l y_0 \sin\frac{m\pi x}{l}dx$$

or

$$A_s = \frac{2}{l}\int_0^l y_0 \sin\frac{s\pi x}{l}dx \tag{4}$$

Now from the diagram

$$\frac{y_0}{k} = \frac{x}{h} \quad if 0 \leq x \leq h$$

$$\frac{y_0}{k} = \frac{l-x}{l-h} \quad if h \leq x \leq l$$

Thus from equation 4

$$\begin{aligned}
A_s &= \frac{2}{l}\left[\int_0^h \frac{kx}{h}\sin\frac{s\pi x}{l}dx + \int_h^l \frac{k(l-x)}{(l-h)}\sin\frac{s\pi x}{l}dx\right] \\
&= \frac{2k}{lh}\left[-x\frac{l}{s\pi}\cos\frac{s\pi x}{l} + \frac{l^2}{s^2\pi^2}\sin\frac{s\pi x}{l}\right]_0^h \\
&+ \frac{2k}{l(l-h)}\left[-(l-x)\frac{l}{s\pi}\cos\frac{s\pi x}{l} - \frac{l^2}{s^2\pi^2}\sin\frac{s\pi x}{l}\right]_h^l \\
&= \frac{2k}{lh}\left[-h\frac{l}{s\pi}\cos\frac{s\pi h}{l} + \frac{l^2}{s^2\pi^2}\sin\frac{s\pi h}{l}\right] \\
&+ \frac{2k}{l(l-h)}\left[-(l-h)\frac{l}{s\pi}\cos\frac{s\pi h}{l} + \frac{l^2}{s^2\pi^2}\sin\frac{s\pi h}{l}\right] \\
&= 2k\left[-\frac{1}{s\pi}\cos\frac{s\pi h}{l} + \frac{l}{hs^2\pi^2}\sin\frac{s\pi h}{l} + \frac{1}{s\pi}\cos\frac{s\pi h}{l} + \frac{l}{(l-h)s^2\pi^2}\sin\frac{s\pi h}{l}\right] \\
&= \frac{2kl}{s^2\pi^2}\left[\sin\frac{s\pi h}{l}\left(\frac{1}{h} + \frac{1}{(l-h)}\right)\right] \\
&= \frac{2kl^2}{h(l-h)\pi^2}\frac{1}{s^2}\sin\frac{s\pi h}{l}
\end{aligned}$$

Hence the general displacement for a plucked string vibration is given by

$$\begin{aligned}
y &= \frac{2kl^2}{h(l-h)\pi^2}\sum_{s=1}^{\infty}\frac{1}{s^2}\sin\frac{s\pi h}{l}\cos\frac{s\pi vt}{l}\sin\frac{s\pi x}{l} \qquad (5) \\
&= \frac{2kl^2}{h(l-h)\pi^2}[\sin\frac{\pi h}{l}\sin\frac{\pi x}{l}\cos\frac{\pi vt}{l} + \frac{1}{4}\sin\frac{2\pi h}{l}\sin\frac{2\pi x}{l}\cos\frac{2\pi vt}{l} \\
&+ \frac{1}{9}\sin\frac{3\pi h}{l}\sin\frac{3\pi x}{l}\cos\frac{3\pi vt}{l}] + ...
\end{aligned}$$

The expression (5) giving the resultant displacement is a rapidly convergent series, since it involves $\frac{1}{s^2}$. Thus effectively only a few harmonics contribute to the resultant vibration.

1.1 Young-Helmholtz's Law

If

$$\sin\frac{s\pi h}{l} = 0, y = 0 \qquad (6)$$

And in that case

$$\sin\frac{s\pi h}{l} = \sin n\pi \qquad (7)$$

where $n = 1, 2, 3.....$ or

$$s = n\frac{l}{h}$$

If

$$\frac{l}{h} = p, s = p, 2p, 3p...... \; etc$$

Thus , in the resultant vibration the p th, $2p$ th, $3p$th ets modes of vibration will be absent.

Young-Helmholtz's law states that those harmonics which have a node at the plucked point will be absent from the resultant vibration of the string.

2 Assignment

Problem: A string is fixed at the points $(0, 0)$ and $(2, 0)$. It is plucked at the midpoint through a height h and then released. Find the general displacement at any point of the string.

Further reading

(1). Principles of acoustics, Basudev Ghosh
(2). Sound, K. Bhattacharyya
(3). Waves and Oscillations, R. N. Chaudhuri